10.2 Graphing Square Root Functions

Essential Question: How can you use transformations of a parent square root function to graph functions of the form $g(x) = a\sqrt{(x-h)} + k$ or $g(x) = \sqrt{\frac{1}{b}(x-h)} + k$?

Resource Locker

Explore Graphing and Analyzing the Parent Square Root Function

Although you have seen how to use imaginary numbers to evaluate square roots of negative numbers, graphing complex numbers and complex valued functions is beyond the scope of this course. For purposes of graphing functions based on the square roots (and in most cases where a square root function is used in a real-world example), the domain and range should both be limited to real numbers.

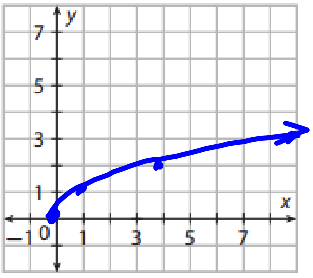
The square root function is the inverse of a quadratic function with a domain limited to positive real numbers. The quadratic function must be a one-to-one function in order to have an inverse, so the domain is limited to one side of the vertex. The square root function is also a one-to-one function as all inverse functions are.

(A) The domain of the square root function (limited to real numbers) is given by $\{x \mid x \geq 0\}$

(B) Fill in the table.

x	$f(x) = \sqrt{x}$
0	0
1	1
4	2
9	3

(C) Plot the points on the graph, and connect them with a smooth curve.



(D) Recall that this function is the inverse of the parent quadratic $\left(f(x) = x^2\right)$ with a domain limited to the nonnegative real numbers. Write the range of this square root function:

$\{y \mid y \geq 0\}$.

(E) The graph appears to be getting flatter as x increases, indicating that the rate of change Decreasing as x increases.

slope

(F) Describe the end behavior of the square root function, $f(x) = \sqrt{x}$. $f(x) \to \infty$ as $x \to \infty$

Reflect

1. **Discussion** Why does the end behavior of the square root function only need to be described at one end?

2. The solution to the equation $x^2 = 4$ is sometimes written as $x = \pm 2$. Explain why the inverse of $f(x) = x^2$ cannot similarly be written as $g(x) = \pm\sqrt{x}$ in order to use all reals as the domain of $f(x)$.

Explore 2 Predicting the Effects of Parameters on the Graphs of Square Root Functions

You have learned how to transform the graph of a function using reflections across the x- and y-axes, vertical and horizontal stretches and compressions, and translations. Here, you will apply those transformations to the graph of the square root function $f(x) = \sqrt{x}$.

When transforming the parent function $f(x) = \sqrt{x}$, you can get functions of the form

$g(x) = a\sqrt{(x-h)} + k$ or $g(x) = \sqrt{\frac{1}{b}(x-h)} + k$.

For each parameter, predict the effect on the graph of the parent function, and then confirm your prediction with a graphing calculator.

(A) Predict the effect of the parameter, h, on the graph of $g(x) = \sqrt{x-h}$ for each function.

a. $g(x) = \sqrt{x-2}$: The graph is a H SHIFT of the graph of $f(x)$ [right/left/up/down] 2 units.

b. $g(x) = \sqrt{x+2}$: The graph is a H SHIFT of the graph of $f(x)$ [right/left/up/down] 2 units.

Check your answers using a graphing calculator.

(B) Predict the effect of the parameter k on the graph of $g(x) = \sqrt{x} + k$ for each function.

a. $g(x) = \sqrt{x} + 2$: The graph is a V SHIFT of the graph of $f(x)$ [right/up/left/down] 2 units.

b. $g(x) = \sqrt{x} - 2$: The graph is a V SHIFT of the graph of $f(x)$ [right/up/left/down] 2 units.

Check your answers using a graphing calculator.

(C) Predict the effect of the parameter a on the graph of $g(x) = a\sqrt{x}$ for each function.

a. $g(x) = 2\sqrt{x}$: The graph is a Vert stretch of the graph of $f(x)$ by a factor of 2.

b. $g(x) = \frac{1}{2}\sqrt{x}$: The graph is a Vert compression of the graph of $f(x)$ by a factor of $\frac{1}{2}$.

c. $g(x) = -\frac{1}{2}\sqrt{x}$: The graph is a ______ compression of the graph of $f(x)$ by a factor of ____ as well as a ______ across the ______.

d. $g(x) = -2\sqrt{x}$: The graph is a ______ stretch of the graph of $f(x)$ by a factor of ____ as well as a ______ across the ______.

Check your answers using a graphing calculator.

(D) Predict the effect of the parameter, b, on the graph of $g(x) = \sqrt{\frac{1}{b}x}$ for each function.

a. $g(x) = \sqrt{\frac{1}{2}x}$: The graph is a HORIZ stretch of the graph of $f(x)$ by a factor of 2.

b. $g(x) = \sqrt{2x}$: The graph is a HORIZ compression of the graph of $f(x)$ by a factor of $\frac{1}{2}$.

c. $g(x) = \sqrt{-\frac{1}{2}x}$: The graph is a ______ stretch of the graph of $f(x)$ by a factor of ____ as well as a ______ across the ______.

d. $g(x) = \sqrt{-2x}$: The graph is a ______ compression of the graph of $f(x)$ by a factor of ____ as well as a ______ across the ______.

Check your answers using a graphing calculator.

Reflect

3. Discussion Describe what the effect of each of the transformation parameters is on the domain and range of the transformed function.

Explain 1 Graphing Square Root Functions

When graphing transformations of the square root function, it is useful to consider the effect of the transformation on two reference points, $(0, 0)$ and $(1, 1)$, that lie on the parent function, and where they map to on the transformed function, $g(x)$.

$f(x) = \sqrt{x}$		$g(x) = a\sqrt{x-h}+k$		$g(x) = \sqrt{\frac{1}{b}(x-h)}+k$	
x	y	x	y	x	y
0	0	h	k	h	k
1	1	$h+1$	$k+a$	$h+b$	$k+1$

The transformed reference points can be found by recognizing that the initial point of the graph is translated from $(0, 0)$ to (h, k). When $g(x)$ involves the parameters a, h, and k, the second transformed reference point is 1 unit to the right of (h, k) and $|a|$ units up or down from (h, k), depending on the sign of a. When $g(x)$ involves the parameters b, h, and k, the second transformed reference point is $|b|$ units left or right from (h, k), depending on the sign of b, and 1 unit above (h, k).

Transformations of the square root function also affect the domain and range. In order to work with real valued inputs and outputs, the domain of the square root function cannot include values of x that result in a negative-valued expression. Negative values of x can be in the domain, as long as they result in nonnegative values of the expression that is inside the square root. Similarly, the value of the square root function is positive by definition, but multiplying the square root function by a negative number, or adding a constant to it changes the range and can result in negative values of the transformed function.

Example 1 **For each of the transformed square root functions, find the transformed reference points and use them to plot the transformed function on the same graph with the parent function. Describe the domain and range using set notation.**

Ⓐ $g(x) = 2\sqrt{x-3} - 2$

$2\boxed{y}-2$

$\boxed{x}+3$

To find the domain:

Square root input must be nonnegative. $x - 3 \geq 0$

Solve the inequality for x. $x \geq 3$

The domain is $\{x \mid x \geq 3\}$.

To find the range:

The square root function is nonnegative. $\sqrt{x-3} \geq 0$

Multiply by 2 $2\sqrt{x-3} \geq 0$

Subtract 2. $2\sqrt{x-3} - 2 \geq -2$

The expression on the left is $g(x)$. $g(x) \geq -2$

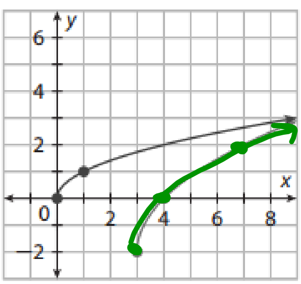
$y \geq -2$

Since $g(x)$ is greater than or equal to -2 for all x in the domain, the range is $\{y \mid y \geq -2\}$.

$(0, 0) \to (3, -2)$

$(1, 1) \to (4, 0)$

$(4, 2) \to (7, 2)$



(B) $g(x) = \sqrt{-\frac{1}{2}(x-2)} + 1$ $\boxed{y}+1$ $-2\boxed{x}+2$

To find the domain:

Square root input must be nonnegative. $-\frac{1}{2}(x-2) \geq \boxed{0}$

Multiply both sides by -2. $x - 2 \boxed{\leq} 0$

Add 2 to both sides. $\boxed{x} \leq 2$

Expressed in set notation, the domain is $\{x \mid \boxed{x \leq 2}\}$.

To find the range:

The square root function is nonnegative. $\sqrt{-\frac{1}{2}(x-2)} \boxed{\geq} 0$

Add 1 both sides $\sqrt{-\frac{1}{2}(x-2)} + 1 \geq \boxed{1}$

Substitute in $\boxed{}$. $g(x) \geq 1$

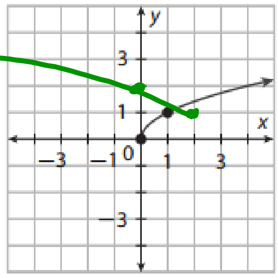
Since $g(x)$ is greater than 1 for all x in the domain, $y \geq 1$

the range (in set notation) is $\{y \mid \boxed{y \geq 1}\}$.

$(0, 0) \rightarrow \boxed{(2, 1)}$

$(1, 1) \rightarrow \boxed{(0, 2)}$

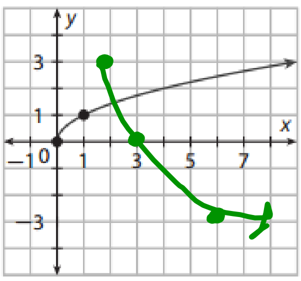
$(4, 2) \rightarrow (-6, 3)$



Your Turn

For each of the transformed square root functions, find the transformed reference points and use them to plot the transformed function on the same graph with the parent function. Describe the domain and range using set notation.

4. $g(x) = -3\sqrt{x-2} + 3$



$(0,0) \rightarrow 2, 3$

$(1,1) \rightarrow 3, 0$

$(4,2) \rightarrow 6, -3$

DOMAIN

$x - 2 \geq 0$

$\{x \mid x \geq 2\}$

$x \geq 2$

$\boxed{x}+2$

$-3\boxed{y}+3$

RANGE

$\sqrt{x-2} \geq 0$

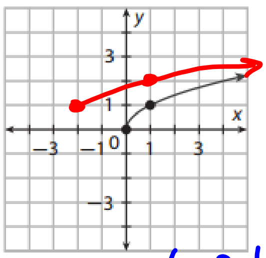
$-3\sqrt{x-2} \leq 0$

$-3\sqrt{x-2}+3 \leq 3$

$y \leq 3$

DOMAIN RANGE

5. $g(x) = \sqrt{\frac{1}{3}(x+2)} + 1$



$(0,0) \rightarrow (-2,1)$
$(1,1) \rightarrow (1,2)$
$(4,2) \rightarrow (10,3)$

$3 \ \boxed{x} - 2$
$\boxed{y} + 1$

Domain:
$\frac{1}{3}(x+2) \geq 0$
$x+2 \geq 0$
$x \geq -2$

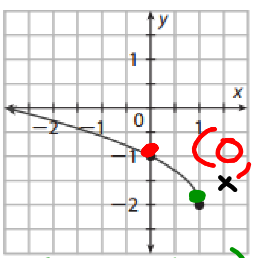
Range:
$\sqrt{x+2} \geq 0$
$\sqrt{\frac{1}{3}(x+2)} \geq 0$
$\sqrt{\frac{1}{3}(x+2)} + 1 \geq 1$
$y \geq 1$

Explain 2 Writing Square Root Functions

Given the graph of a square root function and the form of the transformed function, either $g(x) = a\sqrt{x-h} + k$ or $g(x) = \sqrt{\frac{1}{b}(x-h)} = k$, the transformation parameters can be determined from the transformed reference points. In either case, the initial point will be at (h, k) and readily apparent. The parameter a can be determined by how far up or down the second point (found at $x = h + 1$) is from the initial point, or the parameter b can be determined by how far to the left or right the second point (found at $y = k + 1$) is from the initial point.

Example 2 **Write the function that matches the graph using the indicated transformation format.**

Ⓐ $g(x) = \sqrt{\frac{1}{b}(x-h)} + k$



$(0,-1)$
$x \quad y$

$(h,k) \rightarrow (1,-2)$
$h \quad k$

Initial point: $(h, k) = (1, -2)$

Second point: $(h + b, k + 1) = (0, -1)$

$1 \quad -2$

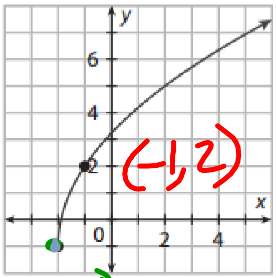
$1 + b = 0$

$b = -1$

The function is $g(x) = \sqrt{-1(x-1)} - 2$.

$-1 = \sqrt{\frac{1}{b}(0-1)} - 2$
$1 = \sqrt{\frac{1}{b}(-1)}$
$1 = \frac{1}{b}(-1)$
$-1 = \frac{1}{b}$
$-b = 1$
$b = -1$

(B) $g(x) = a\sqrt{x-h} + k$



(-2, -1)

Initial point: $(h, k) = (\boxed{-2}, \boxed{-1})$

Second point:

$(h + 1, k + \boxed{A}) = (-1, \boxed{2})$

$\boxed{-1} + a = 2$

$a = \boxed{3}$

The function is $g(x) = \boxed{3}\sqrt{x \boxed{+} 2} - \boxed{1}$.

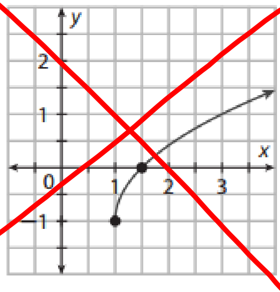
$2 = A\sqrt{-1+2} - 1$

$3 = A$

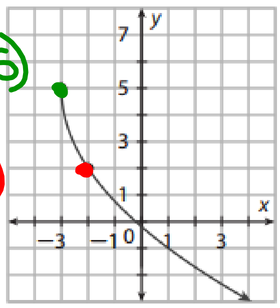
Your Turn

Write the function that matches the graph using the indicated transformation format.

6. $g(x) = \sqrt{\frac{1}{b}(x-h)} + k$



7. $g(x) = a\sqrt{(x-h)} + k$



(-3, 5)

(-2, 2)

$(h+1, k+A)$

$(-3+1, 5+A)$

$(-2, 5+A) \rightarrow (-2, 2)$

$2 = A\sqrt{-2+3} + 5$

$-3 = A$

$5 + A = 2$

$A = -3$

$g(x) = -3\sqrt{x+3} + 5$

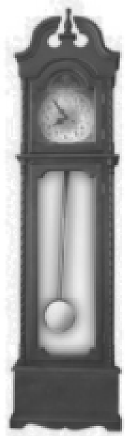
Explain 3 Modeling with Square Root Functions

Square root functions that model real-world situations can be used to investigate average rates of change.

Recall that the average rate of change of the function $f(x)$ over an interval from x_1 to x_2 is given by

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Example 3 **Use a calculator to evaluate the model at the indicated points, and connect the points with a curve to complete the graph of the model. Calculate the average rates of change over the first and last intervals and explain what the rate of change represents.**

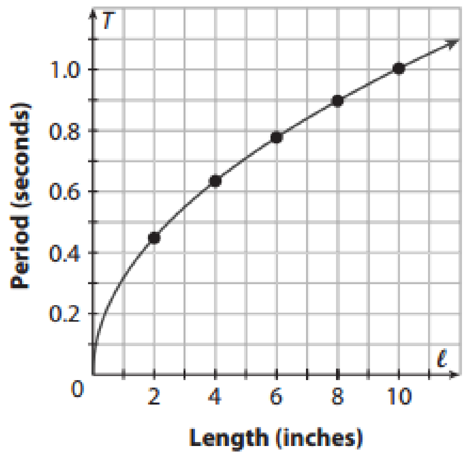


(A) The approximate period T of a pendulum (the time it takes a pendulum to complete one swing) is given in seconds by the formula $T = 0.32\sqrt{\ell}$, where ℓ is the length of the pendulum in inches. Use lengths of 2, 4, 6, 8, and 10 inches.

First find the points for the given x-values.

Length (inches)	Period (seconds)
2	0.45
4	0.64
6	0.78
8	0.91
10	1.01

Plot the points and draw a smooth curve through them.



Find the average increase in period per inch increase in the pendulum length for the first interval and the last interval.

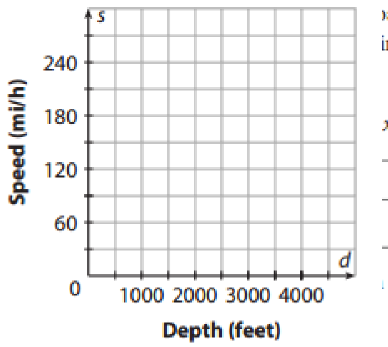
First interval:

$$\text{rate of change} = \frac{0.64 - 0.45}{4 - 2}$$
$$= 0.095$$

Last Interval:

$$\text{rate of change} = \frac{1.01 - 0.91}{10 - 8}$$
$$= 0.05$$

The average rate of change is less for the last interval. The average rate of change represents the increase in pendulum period with each additional inch of length. As the length of the pendulum increases, the increase in period time per inch of length becomes less.

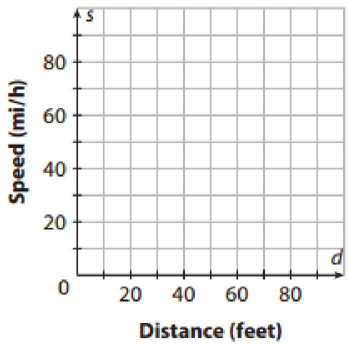


ad. The speed, in miles per hour, from which the in feet, is given by $s(d) = \sqrt{96d}$. Use distances

x-values.

40	60	80	100

curve through them.



rate of change $= \dfrac{\square - \square}{40 - 20}$

$= \square$

Last Interval:

rate of change $= \dfrac{\square - \square}{100 - 80}$

$= \square$

The average rate of change is ______ for the last interval. The average rate of change represents the increase in ________ with each additional ________________. As the available stopping distance increases, the additional increase in speed per foot of stopping distance __________.

Your Turn

Use a calculator to evaluate the model at the indicated points, and connect the points with a curve to complete the graph of the model. Calculate the average rates of change over the first and last intervals and explain what the rate of change represents.

8. The speed in miles per hour of a tsunami can be modeled by the function $s(d) = 3.86\sqrt{d}$, where d is the average depth in feet of the water over which the tsunami travels. Graph this function from depths of 1000 feet to 5000 feet and compare the change in speed with depth from the shallowest interval to the deepest. Use depths of 1000, 2000, 3000, 4000, and 5000 feet for the x-values.

Elaborate

9. What is the difference between the parameters inside the radical (b and h) and the parameters outside the radical (a and k)?

10. Which transformations change the square root function's end behavior?

11. Which transformations change the square root function's initial point location?

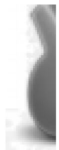
12. Which transformations change the square root function's domain?

13. Which transformations change the square root function's range?

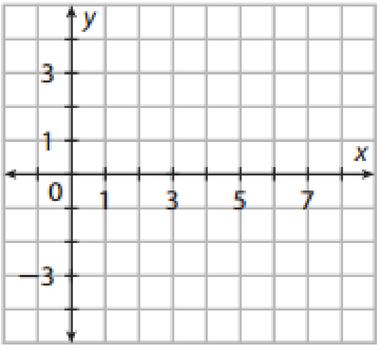
14. **Essential Question Check-In** Describe in your own words the steps you would take to graph a function of the form $g(x) = a\sqrt{x-h} + k$ or $g(x) = \sqrt{\frac{1}{b}(x-h)} + k$ if you were given the values of h and k and using either a or b.

Pg 505 2-4, 6-8, 10, 11, 16

Evaluate: Homework and Practice



1. Graph the functions $f(x) = \sqrt{x}$ and $g(x) = -\sqrt{x}$ on the same grid. Describe the domain, range and end behavior of each function. How are the functions related?



Describe the transformations of $g(x)$ from the parent function $f(x) = \sqrt{x}$.

2. $g(x) = \sqrt{\frac{1}{2}x} + 1$

HORIZ. STRETCH 2
VERT SHIFT UP 1

3. $g(x) = -5\sqrt{x+1} - 3$

H SHIFT left 1
V SHIFT DOWN 3
V STRETCH 5
V REFLECTION

4. $g(x) = \frac{1}{4}\sqrt{x-5} - 2$

H SHIFT R 5
V SHIFT DOWN 2
V STRETCH $\frac{1}{4}$

5. $g(x) = \sqrt{-7(x-7)}$

Describe the domain and range of each function using set notation.

6. $g(x) = \sqrt{\frac{1}{3}(x-1)}$

D: $x \geq 1$
R: $y \geq 0$

$\{x \mid x \geq 1\}$

7. $g(x) = 3\sqrt{x+4} + 3$

D: $x \geq -4$
R: $y \geq 3$

8. $g(x) = \sqrt{-5(x+1)} + 2$

D: $x \leq -1$
R: $y \geq 2$

$-5(x+1) \geq 0$
$x+1 \leq 0$
$x \leq -1$

9. $g(x) = -7\sqrt{x-3} - 5$

Plot the transformed function $g(x)$ on the grid with the parent function, $f(x) = \sqrt{x}$. Describe the domain and range of each function using set notation.

$3\boxed{x}-4$

$\boxed{y}-1$

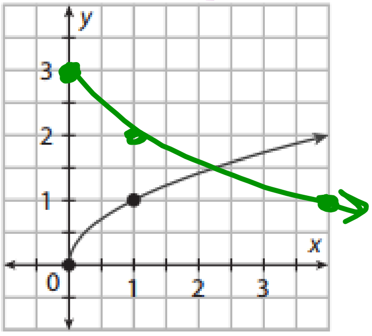
10. $g(x) = -\sqrt{x} + 3$

$\cancel{x}$

$-\boxed{y}+3$

D:
$x \geq 0$

R:
$y \leq 3$



$(0,0) \rightarrow (0, 3)$

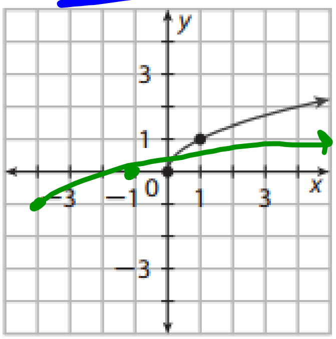
$(1,1) \rightarrow (1, 2)$

$(4,2) \rightarrow (4, 1)$

11. $g(x) = \sqrt{\frac{1}{3}(x+4)} - 1$

D
$x \geq -4$

R:
$y \geq -1$

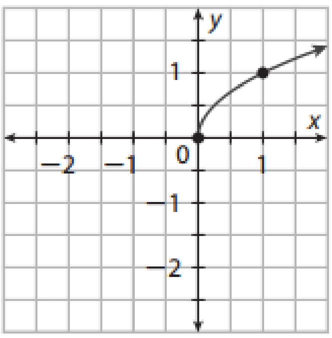


$(0,0) \rightarrow (-4, -1)$

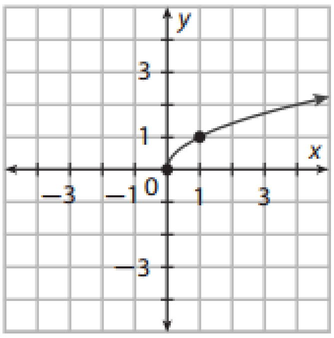
$(1,1) \rightarrow (-1, 0)$

$(4,2) \rightarrow (8, 1)$

12. $g(x) = \sqrt{-\frac{2}{3}\left(x - \frac{1}{2}\right)} - 2$

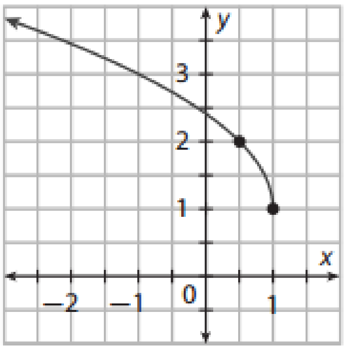


13. $g(x) = 4\sqrt{x+3} - 4$

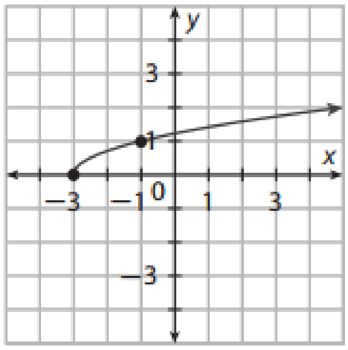


Write the function that matches the graph using the indicated transformation format.

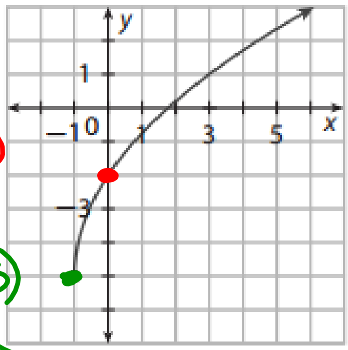
14. $g(x) = \sqrt{\frac{1}{b}(x - h)} + k$



15. $g(x) = \sqrt{\frac{1}{b}(x - h)} + k$



16. $g(x) = a\sqrt{x - h} + k$



x y
(0, -2)

h k
(-1, -5)

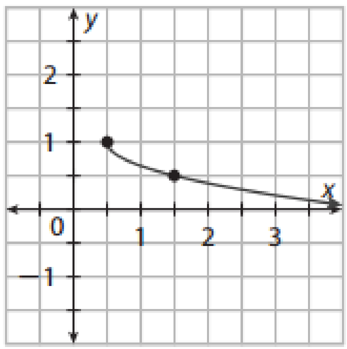
(h+1, k+A)

(0, -5+A) = (0, -2)

-5+A=-2

A=3

17. $g(x) = a\sqrt{x - h} + k$



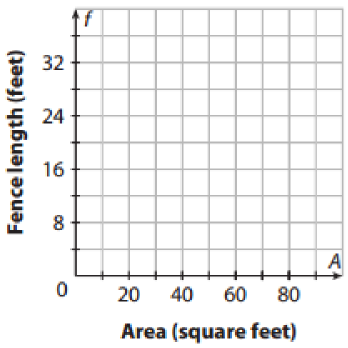
$-2 = A\sqrt{0+1} - 5$

$3 = A$

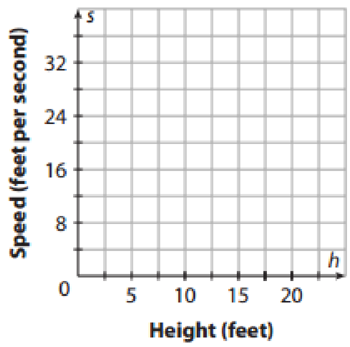
$g(x) = 3\sqrt{x+1} - 5$

Use a calculator to evaluate the model at the indicated points, and connect the points with a curve to complete the graph of the model. Calculate the average rates of change over the first and last intervals and explain what the rate of change represents.

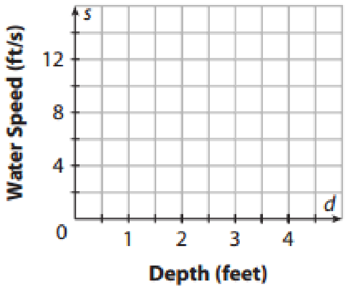
18. A farmer is trying to determine how much fencing to buy to make a square holding pen with a 6-foot gap for a gate. The length of fencing, f, in feet, required as a function of area, A, in square feet, is given by $f(A) = 4\sqrt{A} - 6$. Evaluate the function from 20 ft² to 100 ft² by calculating points every 20 ft².

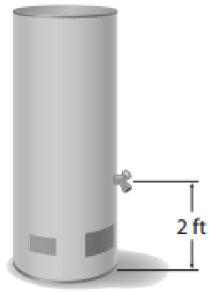


19. The speed, s, in feet per second, of an object dropped from a height, h, in feet, is given by the formula $s(h) = \sqrt{64h}$. Evaluate the function for heights of 0 feet to 25 feet by calculating points every 5 feet.

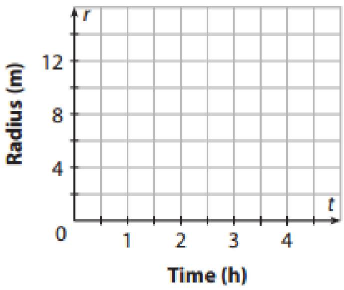


20. Water is draining from a tank at an average speed, s, in feet per second, characterized by the function $s(d) = 8\sqrt{d-2}$, where d is the depth of the water in the tank in feet. Evaluate the function for depths of 2, 3, 4, and 5 feet.





21. A research team studies the effects from an oil spill to develop new methods in oil clean-up. In the spill they are studying, the damaged oil tanker spilled oil into the ocean, forming a roughly circular spill pattern. The spill expanded out from the tanker, increasing the area at a rate of 100 square meters per hour. The radius of the circle is given by the function $r = \sqrt{\frac{100}{\pi}t}$, where t is the time (in hours) after the spill begins. Evaluate the function at hours 0, 1, 2, 3, and 4.



22. Give all of the transformations of the parent function $f(x) = \sqrt{x}$ that result in the function $g(x) = \sqrt{-2(x-3)} + 2$.

A. Horizontal stretch
B. Horizontal compression
C. Horizontal reflection
D. Horizontal translation
E. Vertical stretch
F. Vertical compression
G. Vertical reflection
H. Vertical translation

H.O.T. Focus on Higher Order Thinking

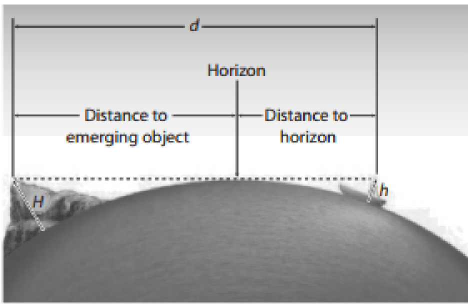
23. Draw Conclusions Describe the transformations to $f(x) = \sqrt{x}$ that result in the function $g(x) = \sqrt{-8x + 16} + 3$.

24. Analyze Relationships Show how a horizontally stretched square root function can sometimes be replaced by a vertical compression by equating the two forms of the transformed square root function.

$$g(x) = a\sqrt{x} = \sqrt{\frac{1}{b}x}$$

What must you assume about a and b for this replacement to result in the same function?

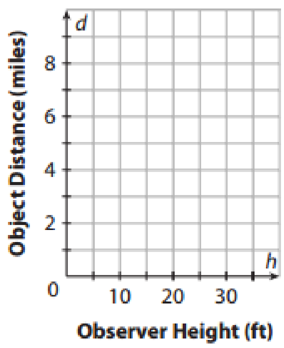
25. **Multi-Step** On a clear day, the view across the ocean is limited by the curvature of Earth. Objects appear to disappear below the horizon as they get farther from an observer. For an observer at height h above the water looking at an object with a height of H (both in feet), the approximate distance (d) in miles at which the object drops below the horizon is given by $d(h) = 1.21\sqrt{h + H}$.



a. What is the effect of the object height, H, on the graph of $d(h)$?

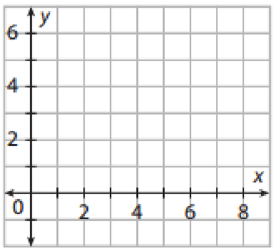
b. What is the domain of the function $d(h)$? Explain your answer.

c. Plot two functions of distance required to see an object over the horizon versus observer height: one for seeing a 2-foot-tall buoy and one for seeing a 20-foot-tall sailboat. Calculate points every 10 feet from 0 to 40 feet.



Lesson Performance Task

With all the coffee beans that come in for processing, a coffee manufacturer cannot sample all of them. Suppose one manufacturer uses the function $s(x) = \sqrt{x} + 1$ to determine how many samples that it must take from x containers in order to obtain a good representative sampling of beans. How does this function relate to the function $f(x) = \sqrt{x}$? Graph both functions. How many samples should be taken from a shipment of 45 containers of beans? Explain why this can only be a whole number answer.



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