18.3 Translating Trigonometric Graphs

Essential Question: How do the constants $h$ and $k$ in the functions $f(x) = a \sin \left( \frac{1}{b} (x - h) \right) + k$ and $f(x) = a \cos \left( \frac{1}{b} (x - h) \right) + k$ affect their graphs?

Explore: Translating the Graph of a Trigonometric Function

In previous lessons, you saw in what ways the graphs of $f(x) = a \sin \left( \frac{1}{b} x \right)$, $f(x) = a \cos \left( \frac{1}{b} x \right)$, and $f(x) = a \tan \left( \frac{1}{b} x \right)$ were vertical and horizontal shrinks and stretches of the graphs of their parent functions. You saw that the vertical stretches and shrinks changed the amplitude of sine and cosine graphs, but did not change the midline on the x-axis, and that the horizontal stretches and compressions changed the period of all of the graphs.

As with other types of functions, you can indicate horizontal and vertical translations in the equations for trigonometric functions. Trigonometric functions in the form $f(x) = a \sin \left( \frac{1}{b} (x - h) \right) + k$, $f(x) = a \cos \left( \frac{1}{b} (x - h) \right) + k$, or $f(x) = a \tan \left( \frac{1}{b} (x - h) \right) + k$ indicate a vertical translation by $k$ and a horizontal translation by $h$.

Answer the following questions about the graph of $f(x) = 0.5 \sin \left( 3 \cdot \frac{x}{2} \right)$.

a. What is the period of the graph?

b. What are the first three x-intercepts for $x \geq 0$?

c. What are the maximum and minimum values of the first cycle for $x \geq 0$, and where do they occur?

d. What are the five key points of the graph that represent the values you found?

Use the key points to sketch one cycle of the graph.
8. Identify $h$ and $k$ for 
$f(x) = 0.5 \sin \left( x - \frac{\pi}{3} \right) + 1.5$, 
and tell what translations they indicate. Find 
the images of the key points of the graph in 
Step A. Finally, sketch the graph from Step A 
again, along with the graph of its image, 
after the indicated translations.

$h = \frac{\pi}{3}$ 
$k = -1.5$

$\text{Shift } \frac{\pi}{3} \quad \text{Shift down } 1.5$

$(0,0) \rightarrow \left( \frac{\pi}{3}, -1.5 \right)$
$(\frac{\pi}{6}, 0) \rightarrow \left( \frac{\pi}{2}, -1 \right)$
$(\frac{\pi}{3}, 0) \rightarrow \left( \frac{2\pi}{3}, -1.5 \right)$
$(\frac{2\pi}{3}, 0) \rightarrow \left( \frac{5\pi}{6}, -2 \right)$

Answer the following questions about the 
graph of $f(x) = \tan \left( \frac{x}{2} \right)$.

a. What is the period of the graph?

b. What is the $x$-intercept of the graph 
at or nearest the origin? What are the 
asymptotes?

c. What are the halfway points on either 
side of the $x$-intercept that you found?

d. What are the three key points of the 
graph that represent the values you 
found?

Use the key points to sketch one cycle of the 
graph. Also show the asymptotes.
Identify $h$ and $k$ for $f(x) = 2\tan\left(\frac{x}{2} + \pi\right) + 3$, and tell what translations they indicate. Find the images of the key points of the graph in Step A, and the new asymptotes. Finally, sketch the graph from Step A again, along with the graph of its image after the indicated translations. (Note: Show the asymptotes for the translated graph, but not for the original graph.)

Reflect

1. Suppose that you are told to extend the graph of the translated function you graphed in Step B to the left and to the right. Without actually drawing the graph, explain how you would do this.

2. What feature of the graphs of the trigonometric functions is represented by the value of the parameter $k$?
**Example 1**

For the function given, identify the period and the midline of the graph, and where the graph crosses the midline. For a sine or cosine function, identify the amplitude and the maximum and minimum values and where they occur. For a tangent function, identify the asymptotes and the values of the half-way points. Then graph one cycle of the function.

\[ f(x) = 3\sin(x - \frac{\pi}{2}) + 1 \]

Period: \( \frac{2\pi}{b} = 1 \), so \( b = 1 \); period = \( 2\pi \cdot 1 = 2\pi \)

Midline: \( y = k \), or \( y = 1 \)

Amplitude: \( a = 3 \)

The point \((0, 0)\) on the graph of the parent function \( y = \sin x \) is translated \( h = \pi \) units to the right and \( k = 1 \) unit up to \((x, 1)\). The graph also crosses the midline at the endpoint of the cycle, \((\pi + 2\pi, 1) = (3\pi, 1)\) and at the point halfway between \((\pi, 1)\) and \((3\pi, 1)\) or at \((2\pi, 1)\). So, the graph contains \((\pi, 1)\), \((2\pi, 1)\) and \((3\pi, 1)\).

Maximum: \( a = 3 \) units above the midline, or \( k + a = 1 + 3 = 4 \); occurs halfway between the first and second midline crossings, or at \( x = \frac{\pi + 2\pi}{2} = \frac{3\pi}{2} \). So, the graph contains \( \left(\frac{3\pi}{2}, 4\right) \).

Minimum: \( a = 3 \) units below the midline, or \( k - a = 1 - 3 = -2 \); occurs halfway between the second and third midline crossings, or at \( x = \frac{2\pi + 3\pi}{2} = \frac{5\pi}{2} \). So, the graph contains \( \left(\frac{5\pi}{2}, -2\right) \).

Plot the key points found and sketch the graph.
$f(x) = -2\tan\frac{1}{2}(x - 2\pi) + 2$

Period: $\frac{1}{b} = \frac{1}{2}$, so $b = 2$; period = $\pi \cdot 2 = 2\pi$

Midline: $y = k$, or $y = 2$

The point $(0, 4)$ on the graph of $y = \tan x$ is translated $h = 2\pi$ units to the right and $k = 2$ units up to $(2\pi, 2)$. There are asymptotes half a cycle to the left and right of this point, or at $x = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$ and $x = 2\pi + \frac{\pi}{2} = \frac{5\pi}{2}$.

Halfway points occur halfway between the asymptotes and where the graph crosses its midline, or at $x = \frac{\pi}{2} + 2\pi = \frac{5\pi}{2}$ and $x = \frac{2\pi}{2} + 3\pi = \frac{5\pi}{2}$.

The halfway points of $y = \tan x$ have $y$-values of $-1$ and $1$. Because $a = \_\_\_\_\_$ in $f(x) = -2\tan\frac{1}{2}(x - 2\pi) + 2$, the halfway points are reflected across the midline and stretched vertically from it by a factor of $\_\_\_\_\_\_\_\_\_$.

The halfway points are:

$\left(\frac{3\pi}{2}, a(-1) + k\right) = \left(\frac{3\pi}{2}, -2(-1) + 2\right) = \left(\frac{3\pi}{2}, 4\right)$ and

$\left(\frac{5\pi}{2}, a(1) + k\right) = \left(\frac{5\pi}{2}, -2(1) + 2\right) = \left(\frac{5\pi}{2}, 0\right)$.

Plot the key points found and sketch the graph.
3. For \( f(x) = 2\cos(x - \frac{\pi}{2}) + 1 \), identify the period, the midline, when the graph crosses the midline, the amplitude, and the maximum and minimum values and where they occur. Then graph one cycle of the function.

**Period:** \( \frac{2\pi}{1} = \pi \)

**Midline:** \( y = 1 \)

**First Point:** \( (0,2) \) + \( (\frac{\pi}{2}, 1) \) \( \rightarrow \) \( (\frac{\pi}{2}, 3) \)

**Crosses Midline:** \( \frac{3\pi}{4}, \frac{5\pi}{4} \)

**Amp:** 2

**Max:** \( (\frac{\pi}{2}, 3) \)

**Min:** \( (\pi, -1) \)

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**Explain 2** Writing General Trigonometric Functions

Because the equations of \( f(x) = a\sin(b(x - h)) + k \), \( f(x) = a\cos(b(x - h)) + k \), or \( f(x) = a\tan(b(x - h)) + k \) directly reflect the physical features of their graphs, it is straightforward to write an equation given a graph of one of these functions.

**Example 2** Write an equation as indicated for the given graph.

\( a \) cosine function

**Period:** \( 2\pi \)

\( \frac{2\pi}{B} = 2\pi \)

\( B = 1 \)

\( \frac{1}{B} = 1 \)

**Midline:** \( y = -2 \)

\( k = -2 \)

\( y = 3\cos(x - \pi) - 2 \)

\( y = -3\cos(x - \pi) - 2 \)

\( y = -3\cos x - 2 \)
Amplitude: \( a = \frac{1 - (-3)}{2} = 3 \)
Midline: \( y = -2 \), so \( k = -2 \).
Period: \( 2\pi \); so, \( 2\pi \cdot b = 2\pi \), and \( b = 1 \).

You can obtain a local maximum at \( x = -\pi \) by translating the graph of \( y = \cos x \) to the left by \( \pi \) units. So, \( h = -\pi \).

A cosine equation is \( f(x) = 3\cos(x + \pi) - 2 \). Notice that the equations \( f(x) = 3\cos(x - \pi) - 2 \) and \( f(x) = -3\cos x - 2 \) also represent the graph.

A tangent function with midline and halfway points shown

Midline: \( y = \frac{1}{2} \), so \( k = \frac{1}{2} \).
Period: \( 2\pi \); so, \( \frac{\pi}{2} \cdot b = 2\pi \), and \( b = 2 \); \( \frac{b}{a} = \frac{1}{2} \).

The halfway points lie \( 3 \) units above and below the midline, so the graph is vertically stretched/shrunk by a factor of \( 3 \). Because the graph is falling from left to right for each cycle, the sign of \( a \) is negative/positive. So, \( a = -3 \).

A cycle of the graph crosses the midline at \( x = \pi \) instead of at \( x = 0 \) as for the graph of \( y = \tan x \), so you can describe the graph by a translation to the right by \( \frac{\pi}{4} \) units. So, \( h = \frac{\pi}{4} \).

A tangent equation is \( f(x) = -3\tan\left(\frac{1}{2}(x + \pi) + \frac{1}{4}\right) \). Notice that the equation \( f(x) = -3\tan\left(\frac{1}{2}(x + \pi) + 1\right) \) also represents the graph.

Reflect

4. How could you write a sine function from the cosine function first described by Example 2A?
5. What is true in general about the graph of a tangent function of the form \( f(x) = a \tan \left( \frac{1}{b}(x - h) \right) + k \) when \( a > 0 \)? when \( a < 0 \)?

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### Your Turn

6. Write a sine function for the graph.

- **Period:** \( \frac{\pi}{2} \)
- \( \frac{2\pi}{B} = \frac{\pi}{2} \)
- \( B = 2 \)
- **Midline:** \( y = -1 \)
- **AMP:** \( 2 \)

\[ Y = 2 \sin \left( \frac{\pi}{2} x \right) - 1 \]

### Explain 3

#### Modeling with General Trigonometric Functions

Many real-world phenomena, such as circular motion and wave motion, involve repeating patterns that are described by trigonometric functions.

#### Example 3

Interpret values of trigonometric models.

**Paddle Wheels** The motion of a point on the outer edge of a riverboat's paddle wheel blade is modeled by \( h(t) = 8 \sin \left( \frac{\pi}{2} (t - 1) \right) + 6 \) where \( h \) is the height in feet measured from the water line and \( t \) is the time in seconds. Identify the period, midline, amplitude, and maximum and minimum values of the graph. For one cycle starting from \( t = 0 \), find all points where the graph intersects its midline and the coordinates of any local maxima and minima. Interpret these points in the context of the problem, and graph one cycle.

Period: \( \frac{1}{b} = \frac{\pi}{2} \), so \( b = \frac{2}{\pi} \), and \( 2\pi \cdot b = 4 \); Midline: \( k = 6 \), so the midline is \( h(t) = 6 \).

Amplitude: \( a = 8 \); Maximum: \( k + a = 6 + 8 = 14 \); Minimum: \( k - a = 6 - 8 = -2 \).

When \( t = 0 \), \( h(t) = 8 \sin \left( \frac{\pi}{2} (0 - 1) \right) + 6 = 8 \sin \left( -\frac{\pi}{2} \right) + 6 = 8(-1) + 6 = -2 \). So, \((0, -2)\) is on the graph. This is a minimum. There is a second minimum at the end of the cycle at \((0 + 4, -2) \equiv (4, -2)\).

A maximum lies halfway between the \( x \)-values of the minima at \( x = \frac{0 + 4}{2} = 2 \). So, \((2, 14)\) is on the graph.
The graph crosses its midline halfway between each local maximum or minimum, or at $x = \frac{0 + 3.5}{2} = 1.75$ and $x = \frac{3.5 + 9.5}{2} = 6.0$. So, $(1.75, -2)$ and $(6.0, 2)$ are on the graph.

**Amusement Parks** The motion of a gondola car on the Ferris wheel at Navy Pier in Chicago can be modeled by $h(t) = 70 \sin \frac{\pi}{2.5} (t - 1.75) + 80$, where $h$ is the height in feet and $t$ is the time in minutes. Identify the period, midline, amplitude, and maximum and minimum values of the graph. For one cycle starting from $t = 0$, find all points where the graph intersects its midline and the coordinates of any local maxima and minima. Interpret these points in the context of the problem, and graph one cycle.

- **Period:** $\frac{1}{b} = \phantom{0}$, so $b = \phantom{0}$, and $2\pi \cdot b = \phantom{0}$;
- **Midline:** $k = \phantom{0}$, so the midline is $h(t) = \phantom{0}$;
- **Amplitude:** $a = \phantom{0}$; Maximum: $k + a = \phantom{0} + \phantom{0} = \phantom{0}$;
- **Minimum:** $k - a = \phantom{0} - \phantom{0} = \phantom{0}$

When $t = 0$, $h(t) = 70 \sin \frac{\pi}{2.5} (0 - 1.75) + 80 = 70 \sin \left( \frac{\pi}{2.5} \right) + 80 = 70 \left( \phantom{0} \right) + 80 = \phantom{0}$. So, $(0, \phantom{0})$ is on the graph. This is a minimum.

There is a second minimum at the end of the cycle at $(0 + \phantom{0}, 10) = \phantom{0}$, 10. A maximum lies halfway between the $x$-values of the minima at $x = \frac{0 + \phantom{0}}{2} = \phantom{0}$. So, $(\phantom{0}, \phantom{0})$ is on the graph.

The graph crosses its midline halfway between each local maximum or
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8. How can being given the first local maximum and local minimum of a cosine function \(a > 0\) help you write its equation?

9. Essential Question Check-In How do positive values of \(h\) affect the graph of a function in the form 
\[ f(x) = a \sin \left( \frac{x}{b} \right) (x - h) + k \] How do negative values of \(k\) affect the graph of this function?

Evaluate: Homework and Practice

For each function, identify the period and the midline of the graph, and where the graph crosses the midline. For a sine or cosine function, identify the amplitude and the maximum and minimum values and where they occur. For a tangent function, identify the asymptotes and the values of the halfway points. Then graph one cycle.

1. \[ f(x) = -3 \sin(x + \pi) + 1 \]

**Period**: \( B = 1 \)
\[ 1.2\pi = 2\pi \]

**Midline**: \( Y = 1 \)

**Start**: \((-\pi, 1)\)

**Crosses Midline**: \(-\pi, 0, \pi\)

**Min**: \((-\frac{3\pi}{2}, -2)\)

**Max**: \((\frac{\pi}{2}, 9)\)

**AMP**: 3
1. \( \frac{A}{b} = 3, \quad b = \frac{4}{3} \quad \text{Period} = 2\pi \cdot \frac{4}{3} = \frac{2\pi}{\frac{3}{2}} \)

2. \( f(x) = 2 \cos x + 1 \)

Start point:
\((0, 2) \rightarrow (0, 1) \rightarrow (0, 3)\)

3. \( f(x) = \tan \left( \frac{1}{2} (x - \pi) + 3 \right) \)

Start point:
\((\pi, 3)\)

Period: \(2\pi\)

Midline: \(y = 3\)
4. \[ f(x) = \sin \left( \frac{\pi}{4}(x - 2) \right) + 3 \]

- **Period**: \( \frac{2\pi}{\pi/4} = 8 \)
- **Midline**: \( y = 3 \)
- **Start Point**: \( (2, 3) \)
- **AMP**: \( 3 \)

\[ \text{Max}: (3, 4) \quad \text{Min}: (5, 0) \]

Write an equation as indicated for the given graph.

5. a sine function
6. a cosine function

7. a tangent function
For the context described, identify the period, midline, amplitude, and maximum and minimum values of the graph. For one cycle starting from $t = 0$, find all points where the graph intersects its midline and the coordinates of any local maxima and minima. Interpret these points in the context of the problem, and graph one cycle.

8. **Historic Technology.** Water turns a water wheel at an old mill. The water comes in at the top of the wheel through a wooden chute. The function $h(t) = 15\cos\frac{\pi}{2}t + 20$ models the height $h$ in feet above the stream into which the water empties of a point on the wheel where $t$ is the time in seconds.

\[ h(t) \]

\[ \begin{array}{c|c}
0 & 2 & 4 & 6 & 8 & 10 & 12 \\
\hline
0 & 20 & 40 & & & & \\
\end{array} \]

9. **Games.** A toy is suspended 36 inches above the floor on a spring. A child reaches up, pulls the toy, and releases it. The function $h(t) = 10\sin\pi(t - 0.5) + 36$ models the toy's height $h$ in inches above the floor after $t$ seconds.

\[ h(t) \]

\[ \begin{array}{c|c}
0 & 0.5 & 1 & 1.5 & 2 \\
\hline
0 & 36 & 48 & & & \\
\end{array} \]
10. Match each sine function with the cosine function that has the same graph.

A. $y = \sin 2x$  
B. $y = \frac{1}{2} \sin 3x$  
C. $y = \frac{1}{2} \sin 4(x - 2)$  
D. $y = \sin (x - \pi) + 1$  
E. $y = \sin \left( x + \frac{\pi}{2} \right) + 4$

1. $y = \cos \left( x + \frac{\pi}{2} \right) + 1$  
2. $y = \frac{1}{2} \cos 4 \left( x - \frac{16 + \pi}{8} \right)$  
3. $y = \cos 2 \left( x - \frac{\pi}{4} \right)$  
4. $y = \cos (x) + 4$  
5. $y = \frac{1}{2} \cos 3 \left( x - \frac{\pi}{6} \right)$

11. How do $h$ and $k$ affect the key points of the graphs of sine, cosine, and tangent functions?

H.O.T. Focus on Higher Order Thinking

12. Explain the Error Sage was told to write the equation of a sine function with a period of $2\pi$, an amplitude of 5, a horizontal translation of 3 units right, and a vertical translation of 6 units up. She wrote the equation $f(x) = 5 \sin 2\pi(x - 3) + 6$. Explain Sage's error and give the correct equation.

13. Critical Thinking Can any sine or cosine function graph be represented by a sine equation with a positive coefficient $a$? Explain your answer.

14. Make a Prediction What will the graph of the function $f(x) = \cos^2 x + \sin^2 x$ look like? Explain your answer and check it on a graphing calculator.
Lesson Performance Task

At a location off a pier on the Maine coastline, the function \( d(t) = 4.36 \cos(0.499t) + 8.79 \) models the depth \( d \) in feet of the water \( t \) hours after the first high tide during a first quarter moon.

a. Identify the amplitude and period of the function as well as the equation of the midline. Describe the graph of this function as a series of transformations of the parent function \( y = \cos x \).

b. Explain how you can use the cosine function from part A to write a sine function to model the depth of the water. Describe the graph of this function as a series of transformations of the parent function \( y = \sin x \).

c. The heights of astronomical tides are affected by the moon phase. A function that models the depth of the water at the same location after the first low tide during a new moon is \( d(t) = 6.31 \sin[0.503(t - 3.13)] + 8.75 \). Identify the amplitude and period of the function as well as the equation of the midline. Describe the graph of this function as a series of transformations of the parent function \( y = \sin x \).

(Continued on next page)
d. Explain how you can use the sine function from part C on the previous page to write a cosine function to model the height of the water. Describe the graph of this function as a series of transformations of the parent function $y = \cos x$.

e. Compare the functions from parts A and B with the functions from parts C and D. How do the tides during a new moon compare to the tides during a first quarter moon?