3.1 Solving Quadratic Equations by Taking Square Roots

Essential Question: What is an imaginary number, and how is it useful in solving quadratic equations?

Explore Investigating Ways of Solving Simple Quadratic Equations

There are many ways to solve a quadratic equation. Here, you will use three methods to solve the equation \( x^2 = 16 \): by graphing, by factoring, and by taking square roots.

A Solve \( x^2 = 16 \) by graphing.

First treat each side of the equation as a function, and graph the two functions, which in this case are \( f(x) = x^2 \) and \( g(x) = 16 \), on the same coordinate plane.

Then identify the \( x \)-coordinates of the points where the two graphs intersect.

\[ x = \square \] or \( x = \square \)

B Solve \( x^2 = 16 \) by factoring.

This method involves rewriting the equation so that 0 is on one side in order to use the zero-product property, which says that the product of two numbers is 0 if and only if at least one of the numbers is 0.

Write the equation.

\[ x^2 = 16 \]

Subtract 16 from both sides.

\[ x^2 - \square = 0 \]

Factor the difference of two squares.

\[ (x + \square)(x - 4) = 0 \]

Apply the zero-product property.

\[ x + \square = 0 \quad \text{or} \quad x - 4 = 0 \]

Solve for \( x \).

\[ x = \square \] or \( x = 4 \)

C Solve \( x^2 = 16 \) by taking square roots.

A real number \( x \) is a square root of a nonnegative real number \( a \) provided \( x^2 = a \). A square root is written using the radical symbol \( \sqrt{\square} \). Every positive real number \( a \) has both a positive square root, written \( +\sqrt{a} \), and a negative square root, written \( -\sqrt{a} \). For instance, the square roots of 9 are \( \pm\sqrt{9} \) (read “plus or minus the square root of 9”), or \( \pm 3 \). The number 0 has only itself as its square root: \( \pm\sqrt{0} = 0 \).

Write the equation.

\[ x^2 = 16 \]

Use the definition of square root.

\[ x = \pm\sqrt{16} \]

Simplify the square roots.

\[ x = \square \]
Reflect

1. Which of the three methods would you use to solve \( x^2 = 5 \)? Explain, and then use the method to find the solutions.

________________________________________________________________________
________________________________________________________________________
________________________________________________________________________

2. Can the equation \( x^2 = -9 \) be solved by any of the three methods? Explain.

________________________________________________________________________
________________________________________________________________________
________________________________________________________________________

**Explain 1  Finding Real Solutions of Simple Quadratic Equations**

When solving a quadratic equation of the form \( ax^2 + c = 0 \) by taking square roots, you may need to use the following properties of square roots to simplify the solutions. (In a later lesson, these properties are stated in a more general form and then proved.)

<table>
<thead>
<tr>
<th>Property Name</th>
<th>Words</th>
<th>Symbols</th>
<th>Numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product property of square roots</td>
<td>The square root of a product equals the product of the square roots of the factors.</td>
<td>( \sqrt{ab} = \sqrt{a} \cdot \sqrt{b} ) where ( a \geq 0 ) and ( b \geq 0 )</td>
<td>( \sqrt{12} = \sqrt{4} \cdot \sqrt{3} ) = ( \frac{2\sqrt{3}}{1} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quotient property of square roots</td>
<td>The square root of a fraction equals the quotient of the square roots of the numerator and the denominator.</td>
<td>( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} ) where ( a \geq 0 ) and ( b &gt; 0 )</td>
<td>( \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{\sqrt{9}} ) = ( \frac{\sqrt{5}}{3} )</td>
</tr>
</tbody>
</table>

Using the quotient property of square roots may require an additional step of rationalizing the denominator if the denominator is not a rational number. For instance, the quotient property allows you to write \( \sqrt{\frac{7}{9}} \) as \( \frac{\sqrt{7}}{\sqrt{9}} \) but \( \sqrt{7} \) is not a rational number. To rationalize the denominator, multiply \( \frac{\sqrt{7}}{\sqrt{9}} \) by \( \frac{\sqrt{9}}{\sqrt{9}} \) (a form of 1) and get this result: \( \frac{\sqrt{7} \cdot \sqrt{9}}{\sqrt{9} \cdot \sqrt{9}} = \frac{\sqrt{63}}{9} = \frac{\sqrt{7} \cdot 9}{9^2} = \frac{\sqrt{7}}{9} \).
Example 1  Solve the quadratic equation by taking square roots.

A  
\[2x^2 - 16 = 0\]

Add 16 to both sides.  \[2x^2 = 16\]
Divide both sides by 2.  \[x^2 = 8\]
Use the definition of square root.  \[x = \pm \sqrt{8}\]
Use the product property.  \[x = \pm \sqrt{4} \cdot \sqrt{2}\]
Simplify.  \[x = \pm 2\sqrt{2}\]

B  
\[-5x^2 + 9 = 0\]

Subtract 9 from both sides.  \[-5x^2 = -9\]
Divide both sides by \[\frac{5}{5}\].  \[x^2 = \frac{9}{5}\]
Use the definition of square root.  \[x = \pm \sqrt{\frac{9}{5}}\]
Use the quotient property.  \[x = \pm \frac{3}{\sqrt{5}}\]
Simplify the numerator.  \[x = \pm \frac{3}{\sqrt{5}}\times \frac{\sqrt{5}}{\sqrt{5}}\]
Rationalize the denominator.

Your Turn  
Solve the quadratic equation by taking square roots.

3.  \[x^2 - 24 = 0\]

\[x^2 = 24\]
\[x = \pm 2\sqrt{6}\]

4.  \[-4x^2 + 13 = 0\]

\[x = \frac{1}{2}\sqrt{13}\]
\[x = -\frac{1}{2}\sqrt{13}\]

Explain 2  Solving a Real-World Problem Using a Simple Quadratic Equation

Two commonly used quadratic models for falling objects near Earth’s surface are the following:

\[\text{Distance fallen (in feet) at time } t \text{ (in seconds): } d(t) = 16t^2\]

\[\text{Height (in feet) at time } t \text{ (in seconds): } h(t) = h_0 - 16t^2 \text{ where } h_0 \text{ is the object’s initial height (in feet)}\]

For both models, time is measured from the moment that the object begins to fall. A negative value of \(t\) would represent a time before the object began falling, so negative values of \(t\) are excluded from the domains of these functions. This means that for any equation of the form \(d(t) = c\) or \(h(t) = c\) where \(c\) is a constant, a negative solution should be rejected.
Example 2  Write and solve an equation to answer the question. Give the exact answer and, if it's irrational, a decimal approximation (to the nearest tenth of a second).

A) If you drop a water balloon, how long does it take to fall 4 feet?
Using the model \( d(t) = 16t^2 \), solve the equation \( d(t) = 4 \).

Write the equation.  \( 16t^2 = 4 \)

Divide both sides by 16.  \( t^2 = \frac{1}{4} \)

Use the definition of square root.  \( t = \pm \sqrt{\frac{1}{4}} \)

Use the quotient property.  \( t = \pm \frac{1}{2} \)

Reject the negative value of \( t \). The water balloon falls 4 feet in \( \frac{1}{2} \) second.

B) The rooftop of a 5-story building is 50 feet above the ground. How long does it take the water balloon dropped from the rooftop to pass by a third-story window at 24 feet?
Using the model \( h(t) = h_0 - 16t^2 \), solve the equation \( h(t) = 24 \). (When you reach the step at which you divide both sides by \(-16\), leave 16 in the denominator rather than simplifying the fraction because you'll get a rational denominator when you later use the quotient property.)

Write the equation.  \( 50 - 16t^2 = 24 \)

Subtract 50 from both sides.  \( -16t^2 = -26 \)

Divide both sides by \(-16\).  \( t^2 = \frac{16}{26} \)

Use the definition of square root.  \( t = \pm \sqrt{\frac{16}{26}} \)

Use the quotient property to simplify.  \( t = \pm \frac{4}{\sqrt{26}} \)

Reject the negative value of \( t \). The water balloon passes by the third-story window in \( \frac{4}{\sqrt{26}} \) seconds.

Reflect

5. Discussion Explain how the model \( h(t) = h_0 - 16t^2 \) is built from the model \( d(t) = 16t^2 \).
Your Turn

Write and solve an equation to answer the question. Give the exact answer and, if it’s irrational, a decimal approximation (to the nearest tenth of a second).

6. How long does it take the water balloon described in Part B to hit the ground?

7. On the moon, the distance \( d \) (in feet) that an object falls in time \( t \) (in seconds) is modeled by the function \( d(t) = \frac{1}{2}gt^2 \). Suppose an astronaut on the moon drops a tool. How long does it take the tool to fall 4 feet?

---

**Explain 3** Defining Imaginary Numbers

You know that the quadratic equation \( x^2 = 1 \) has two real solutions, the equation \( x^2 = 0 \) has one real solution, and the equation \( x^2 = -1 \) has no real solutions. By creating a new type of number called imaginary numbers, mathematicians allowed for solutions of equations like \( x^2 = -1 \).

*Imaginary numbers* are the square roots of negative numbers. These numbers can all be written in the form \( bi \) where \( b \) is a nonzero real number and \( i \), called the imaginary unit, represents \( \sqrt{-1} \). Some examples of imaginary numbers are the following:

- \( 2i \)
- \( -5i \)
- \( \frac{i}{3} \) or \( -\frac{1}{3}i \)
- \( i\sqrt{2} \) (Write the \( i \) in front of the radical symbol for clarity.)
- \( \frac{i\sqrt{3}}{2} \) or \( \frac{\sqrt{3}}{2}i \)

Given that \( i = \sqrt{-1} \), you can conclude that \( i^2 = -1 \). This means that the square of any imaginary number is a negative real number. When squaring an imaginary number, use the power of a product property of exponents: \((ab)^m = a^m \cdot b^m\).
Example 3  Find the square of the imaginary number.

A  \(5i\)

\[(5i)^2 = 5^2 \cdot i^2 = 25(-1) = -25\]

B  \(-i\sqrt{2}\)

\[(-i\sqrt{2})^2 = \left(-\sqrt{\frac{2}{3}}\right)^2 = 2(-1) = -2\]

Reflect

8. By definition, \(i\) is a square root of \(-1\). Does \(-1\) have another square root? Explain.

Your Turn

Find the square of the imaginary number.

9. \(-2i\)

\[(-2i)^2 = 4(-1) = -4\]

10. \(\frac{\sqrt{3}}{3}i\)

\[\left(\frac{\sqrt{3}}{3}i\right)^2 = -\frac{1}{3}\]

Explain 4  Finding Imaginary Solutions of Simple Quadratic Equations

Using imaginary numbers, you can solve simple quadratic equations that do not have real solutions.

Example 4  Solve the quadratic equation by taking square roots. Allow for imaginary solutions.

\[x^2 + 12 = 0\]

Write the equation.

Subtract 12 from both sides.

Use the definition of square root.

Use the product property.

\[x = \pm \sqrt{-12} = \pm 2\sqrt{3}\]

\[x = \pm \sqrt{4(-1)(3)} = \pm 2\sqrt{3}\]
3.1 Solving Quadratic Equations by Taking Square Roots

8. \(4x^2 + 11 = 6\)

Write the equation.
Subtract 11 from both sides.
Divide both sides by \(4\).
Use the definition of square root.
Use the quotient property.

\[
x^2 = \frac{-5}{4}
\]

\[
x = \pm \sqrt{\frac{-5}{4}} = \pm \frac{i\sqrt{5}}{2}
\]

Your Turn

Solve the quadratic equation by taking square roots. Allow for imaginary solutions.

11. \(\frac{1}{4}x^2 + 9 = 0\)

\[
x = \pm 6i
\]

12. \(-5x^2 + 3 = 10\)

\[
x = \pm \frac{i\sqrt{13}}{5}
\]

Elaborate

13. The quadratic equations \(4x^2 + 32 = 0\) and \(4x^2 - 32 = 0\) differ only by the sign of the constant term. Without actually solving the equations, what can you say about the relationship between their solutions?

14. What kind of a number is the square of an imaginary number?

15. Why do you reject negative values of \(t\) when solving equations based on the models for a falling object near Earth's surface, \(d(t) = 16t^2\) for distance fallen and \(h(t) = h_0 - 16t^2\) for height during a fall?

16. Essential Question Check-In Describe how to find the square roots of a negative number.
Evaluate: Homework and Practice

1. Solve the equation \( x^2 - 2 = 7 \) using the indicated method.
   a. Solve by graphing.
      \[
      (0,0) \rightarrow (0, -2) \\
      (1, 1) \rightarrow (1, -1) \\
      (-1, 1) \rightarrow (-1, -1) \\
      (2, 4) \rightarrow (2, 2) \\
      (-2, 4) \rightarrow (-2, 2) \\
      (3, 9) \rightarrow (3, -7)
      \]
      \[x = -3\]
   b. Solve by factoring.
      \[
      x^2 - 2 = 7 \\
      x^2 - 9 = 0 \\
      (x + 3)(x - 3) = 0 \\
      x + 3 = 0 \quad x - 3 = 0 \\
      x = -3 \quad x = 3
      \]
   c. Solve by taking square roots.
      \[
      x^2 - 2 = 7 \\
      x^2 = 9 \\
      x = \pm \sqrt{9} \\
      x = \pm 3
      \]

2. Solve the equation \( 2x^2 + 3 = 5 \) using the indicated method.
   a. Solve by graphing.
      \[
      x = -1, 1
      \]
   b. Solve by factoring.
      \[
      2x^2 + 3 = 5 \\
      2x^2 - 2 = 0 \\
      2(x^2 - 1) = 0 \\
      (x + 1)(x - 1) = 0 \\
      x + 1 = 0 \quad x - 1 = 0 \\
      x = -1 \quad x = 1
      \]
   c. Solve by taking square roots.
      \[
      2x^2 + 3 = 5 \\
      2x^2 = 2 \\
      x^2 = 1 \\
      x = \pm \sqrt{1} \\
      x = \pm 1
      \]
Solve the quadratic equation by taking square roots.

3. \[ 4x^2 = 24 \]

4. \[ -\frac{x^2}{2} + 15 = 0 \]

5. \[ 2(5 - 5x^2) = 5 \]

6. \[ 3x^2 - 8 = 12 \]

Write and solve an equation to answer the question. Give the exact answer and, if it’s irrational, a decimal approximation (to the nearest tenth of a second).

7. A squirrel in a tree drops an acorn. How long does it take the acorn to fall 20 feet?

\[ D(t) = 16t^2 \]
\[ 20 = 16t^2 \]
\[ \frac{20}{16} = t^2 \]
\[ \frac{20}{16} = \frac{5}{4} \]
\[ \sqrt{\frac{5}{4}} = t \]
\[ \frac{\sqrt{5}}{2} \]

1.1 sec

8. A person washing the windows of an office building drops a squeegee from a height of 60 feet. How long does it take the squeegee to pass by another window washer working at a height of 20 feet?
3.1 Solving Quadratic Equations by Taking Square Roots

Geometry: Determine the lengths of the sides of the rectangle using the given area. Give answers both exactly and approximately (to the nearest tenth).

9. The area of the rectangle is 45 cm².

10. The area of the rectangle is 54 cm².

Find the square of the imaginary number.

11. \(3^2 \cdot i^2\)

12. \(i\sqrt{5}\)

13. \(-\frac{i\sqrt{2}}{2}\) \(\cdot 1^2\)

Determine whether the quadratic equation has real solutions or imaginary solutions by solving the equation.

14. \(15x^2 - 10 = 0\)

15. \(\frac{1}{2}x^2 + 12 = 4\)

16. \(5(2x^2 - 3) = 4(x^2 - 10)\)

\[
15x^2 = 10 \quad \frac{-12}{12} \quad \frac{1}{2}x^2 = -8
\]

\[
x^2 = \frac{10}{15} \quad x^2 = -16
\]

\[
x^2 = \frac{2}{3} \quad x = \pm \frac{\sqrt{-16}}{\sqrt{3}}
\]

\[
x = \pm \frac{4i}{\sqrt{3}} \quad \text{Imaginary}
\]

\[
x = -\frac{\sqrt{3} \cdot 16}{3} \quad \text{Real}
\]
Solve the quadratic equation by taking square roots. Allow for imaginary solutions.

17. \( x^2 = -81 \) \( x = \pm \sqrt{-81} \) \( x = \pm 9i \)

18. \( x^2 + 64 = 0 \)

19. \( 5x^2 - 4 = -8 \)

20. \( 7x^2 + 10 = 0 \)

**Geometry** Determine the length of the sides of each square using the given information. Give answers both exactly and approximately (to the nearest tenth).

21. The area of the larger square is 42 cm² more than the area of the smaller square.

22. If the area of the larger square is decreased by 28 cm², the result is half of the area of the smaller square.
23. Determine whether each of the following numbers is real or imaginary.
   a. \( i \)  
      \[ \checkmark \text{Real} \] \[ \checkmark \text{Imaginary} \]
   b. A square root of 5  
      \[ \checkmark \text{Real} \] \[ \checkmark \text{Imaginary} \]
   c. \((2i)^2 = 4i^2 = -4\)  
      \[ \checkmark \text{Real} \] \[ \checkmark \text{Imaginary} \]
   d. \((-5)^3 = -125\)  
      \[ \checkmark \text{Real} \] \[ \checkmark \text{Imaginary} \]
   e. \(\sqrt{-3}\)  
      \[ \checkmark \text{Real} \] \[ \checkmark \text{Imaginary} \]
   f. \(-\sqrt{10}\)  
      \[ \checkmark \text{Real} \] \[ \checkmark \text{Imaginary} \]

H.O.T. Focus on Higher Order Thinking

24. Critical Thinking  
When a batter hits a baseball, you can model the ball's height using a quadratic function that accounts for the ball's initial vertical velocity. However, once the ball reaches its maximum height, its vertical velocity is momentarily 0 feet per second, and you can use the model \( h(t) = -16t^2 + h_o \) to find the ball's height \( h \) (in feet) at time \( t \) (in seconds) as it falls to the ground.

a. Suppose a fly ball reaches a maximum height of 67 feet and an outfielder catches the ball 3 feet above the ground. How long after the ball begins to descend does the outfielder catch the ball?
   \[ 3 = 67 - 16t^2 \]
   \[ -64 = -16t^2 \]
   \[ t^2 = \frac{64}{16} \]
   \[ t = \sqrt{4} = 2 \text{ sec} \]

b. Can you determine (without writing or solving any equations) the total time the ball was in the air? Explain your reasoning and state any assumptions you make.