5.1 Graphing Cubic Functions

Essential Question: How are the graphs of \( f(x) = a(x-h)^3 + k \) and \( f(x) = \left(\frac{1}{b}(x-h)\right)^3 + k \) related to the graph of \( f(x) = x^3 \)?

Explore 1 Graphing and Analyzing \( f(x) = x^3 \)

You know that a quadratic function has the standard form \( f(x) = ax^2 + bx + c \) where \( a, b, \) and \( c \) are real numbers and \( a \neq 0 \). Similarly, a cubic function has the standard form \( f(x) = ax^3 + bx^2 + cx + d \) where \( a, b, c \) and \( d \) are all real numbers and \( a \neq 0 \). You can use the basic cubic function, \( f(x) = x^3 \), as the parent function for a family of cubic functions related through transformations of the graph of \( f(x) = x^3 \).

A. Complete the table, graph the ordered pairs, and then draw a smooth curve through the plotted points to obtain the graph of \( f(x) = x^3 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-8</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

B. Use the graph to analyze the function and complete the table.

<table>
<thead>
<tr>
<th>Attributes of ( f(x) = x^3 )</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Domain</td>
<td>( \mathbb{R} )</td>
</tr>
<tr>
<td>Range</td>
<td>( \mathbb{R} )</td>
</tr>
<tr>
<td>End behavior</td>
<td>( \lim_{x \to +\infty} f(x) = \infty ) ( \lim_{x \to -\infty} f(x) = -\infty )</td>
</tr>
<tr>
<td>Zeros of the function</td>
<td>( x = 0 )</td>
</tr>
<tr>
<td>Where the function has positive values</td>
<td>( x &gt; 0 )</td>
</tr>
<tr>
<td>Where the function has negative values</td>
<td>( x &lt; 0 )</td>
</tr>
<tr>
<td>Where the function is increasing</td>
<td>Always</td>
</tr>
<tr>
<td>Where the function is decreasing</td>
<td>The function never decreases.</td>
</tr>
<tr>
<td>Is the function even, odd, or neither?</td>
<td>Odd, because ( f(-x) = -f(x) )</td>
</tr>
</tbody>
</table>

\[ f(x) = x^3 \]
\[ f(-x) = (-x)^3 \]
\[ -f(x) = -x^3 \]
Reflect

1. How would you characterize the rate of change of the function on the intervals \([-1, 0]\) and \([0, 1]\) compared with the rate of change on the intervals \([-2, -1]\) and \([1, 2]\)? Explain.

2. A graph is said to be symmetric about the origin (and the origin is called the graph's point of symmetry) if for every point \((x, y)\) on the graph, the point \((-x, -y)\) is also on the graph. Is the graph of \(f(x) = x^3\) symmetric about the origin? Explain.

3. The graph of \(g(x) = (-x)^3\) is a reflection of the graph of \(f(x) = x^3\) across the \(y\)-axis, while the graph of \(h(x) = -x^3\) is a reflection of the graph of \(f(x) = x^3\) across the \(x\)-axis. If you graph \(g(x)\) and \(h(x)\) on a graphing calculator, what do you notice? Explain why this happens.

Explain 1  Graphing Combined Transformations of \(f(x) = x^3\)

When graphing transformations of \(f(x) = x^3\), it helps to consider the effect of the transformations on the three reference points on the graph of \(f(x): (-1, -1), (0, 0),\) and \((1, 1)\). The table lists the three points and the corresponding points on the graph of \(g(x) = a\left(\frac{1}{b}(x - h)\right)^3 + k\). Notice that the point \((0, 0)\), which is the point of symmetry for the graph of \(f(x)\), is affected only by the parameters \(h\) and \(k\). The other two reference points are affected by all four parameters.

<table>
<thead>
<tr>
<th>(f(x) = x^3)</th>
<th>(g(x) = a\left(\frac{1}{b}(x - h)\right)^3 + k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>(y)</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>
Example 1  Identify the transformations of the graph of \( f(x) = x^3 \) that produce the graph of the given function \( g(x) \). Then graph \( g(x) \) on the same coordinate plane as the graph of \( f(x) \) by applying the transformations to the reference points \((-1, -1), (0, 0), \text{ and } (1, 1)\).

\[ g(x) = 2(x - 1)^3 - 1 \]

The transformations of the graph of \( f(x) \) that produce the graph of \( g(x) \) are:

- a vertical stretch by a factor of 2
- a translation of 1 unit to the right and 1 unit down

Note that the translation of 1 unit to the right affects only the \( x \)-coordinates of points on the graph of \( f(x) \), while the vertical stretch by a factor of 2 and the translation of 1 unit down affect only the \( y \)-coordinates.

<table>
<thead>
<tr>
<th>( f(x) = x^3 )</th>
<th>( g(x) = 2(x - 1)^3 - 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>(1)</td>
<td>(1)</td>
</tr>
</tbody>
</table>

\[ g(x) = \left(2(x + 3)\right)^3 + 4 \]

The transformations of the graph of \( f(x) \) that produce the graph of \( g(x) \) are:

- a horizontal compression by a factor of \( \frac{1}{2} \)
- a translation of 3 units to the left and 4 units up

Note that the horizontal compression by a factor of \( \frac{1}{2} \) and the translation of 3 units to the left affect only the \( x \)-coordinates of points on the graph of \( f(x) \), while the translation of 4 units up affects only the \( y \)-coordinates.

<table>
<thead>
<tr>
<th>( f(x) = x^3 )</th>
<th>( g(x) = \left(2(x + 3)\right)^3 + 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-1)</td>
</tr>
<tr>
<td>(0)</td>
<td>(0)</td>
</tr>
<tr>
<td>(1)</td>
<td>(1)</td>
</tr>
</tbody>
</table>
Identify the transformations of the graph of \( f(x) = x^3 \) that produce the graph of the given function \( g(x) \). Then graph \( g(x) \) on the same coordinate plane as the graph of \( f(x) \) by applying the transformations to the reference points \((-1, -1), (0, 0), \) and \((1, 1)\).

\[
g(x) = \frac{1}{2}(x - 3)^3
\]

- **Horizontal Shift Right 3**
- **Vertical Stretch \( \frac{1}{2} \)**
- **Reflects Over X-Axis**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = x^3 )</th>
<th>( y = \frac{1}{2}(x - 3)^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1.5</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1.5</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>64</td>
<td>-1.5</td>
</tr>
</tbody>
</table>

**Explain 2** Writing Equations for Combined Transformations of \( f(x) = x^3 \)

Given the graph of the transformed function \( g(x) = a\left(\frac{1}{2}(x - h)\right) + k \), you can determine the values of the parameters by using the same reference points that you used in graphing \( g(x) \) in the previous example.

**Example 2** A general equation for a cubic function \( g(x) \) is given along with the function's graph. Write a specific equation by identifying the values of the parameters from the reference points shown on the graph.

\[
g(x) = a(x - h)^3 + k
\]

Identify the values of \( h \) and \( k \) from the point of symmetry.

\((h, k) = (2, 1)\), so \( h = 2 \) and \( k = 1 \).

Identify the value of \( a \) from either of the other two reference points.

The **uppermost** reference point has general coordinates \((h + 1, a + k)\).
Substituting 2 for \( h \) and 1 for \( k \) and setting the general coordinates equal to the actual coordinates gives this result:

\((h + 1, a + k) = (3, a + 1) = (3, 4)\), so \( a = 3 \).

Write the function using the values of the parameters: \( g(x) = 3(x - 2)^3 + 1 \)
5.1 Graphing Cubic Functions

8. \( g(x) = \left( \frac{1}{b^3} - h \right)^3 + k \)

Identify the values of \( h \) and \( k \) from the point of symmetry.

\( (h, k) = (-4, 1) \), so \( h = -4 \) and \( k = 1 \).

Identify the value of \( b \) from either of the other two reference points.

The rightmost reference point has general coordinates

\( (b + h, 1 + k) \). Substituting \(-4\) for \( h \) and \( \frac{1}{b} \) for \( k \) and setting the general coordinates equal to the actual coordinates gives this result:

\( (b + h, 1 + k) = \left( b - 4, \frac{1}{b} \right) = (-3.5, 2) \), so \( b = \frac{1}{5} \).

Write the function using the values of the parameters, and then simplify.

\[ g(x) = \left( \frac{1}{\frac{1}{5}} \right) \left( x - (-4) \right)^3 + 1 \]

or

\[ g(x) = \left( \frac{5}{4} \right) (x + 4)^3 + 1 \]

Your Turn

A general equation for a cubic function \( g(x) \) is given along with the function’s graph. Write a specific equation by identifying the values of the parameters from the reference points shown on the graph.

5. \( g(x) = a(x - h)^3 + k \)

6. \( g(x) = \left( \frac{1}{b(x - h)} \right)^3 + k \)

\[ a + k = 1 \]
\[ a - 2 = 1 \]
\[ a = 3 \]

\[ g(x) = 3(x + 2)^3 - 2 \]

\[ b + h = 5 \]
\[ b + 1 = 5 \]
\[ b = 4 \]

\[ g(x) = \left( \frac{1}{4} (x - 1) \right)^3 - 1 \]
Modeling with a Transformation of $f(x) = x^3$

You may be able to model a real-world situation that involves volume with a cubic function. Sometimes mass may also be involved in the problem. Mass and volume are related through density, which is defined as an object’s mass per unit volume. If an object has mass $m$ and volume $V$, then its density $d$ is $d = \frac{m}{V}$. You can rewrite the formula as $m = dV$ to express mass in terms of density and volume.

Example 3

Use a cubic function to model the situation, and graph the function using calculated values of the function. Then use the graph to obtain the indicated estimate.

Estimate the length of an edge of a child’s alphabet block (a cube) that has a mass of 23 g and is made from oak with a density of 0.72 g/cm$^3$.

Let $\ell$ represent the length (in centimeters) of an edge of the block. Since the block is a cube, the volume $V$ (in cubic centimeters) is $V(\ell) = \ell^3$. The mass $m$ (in grams) of the block is $m(\ell) = 0.72 \cdot V(\ell) = 0.72\ell^3$. Make a table of values for this function.

<table>
<thead>
<tr>
<th>Length (cm)</th>
<th>Mass (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.72</td>
</tr>
<tr>
<td>2</td>
<td>5.76</td>
</tr>
<tr>
<td>3</td>
<td>19.44</td>
</tr>
<tr>
<td>4</td>
<td>46.08</td>
</tr>
</tbody>
</table>

Draw the graph of the mass function, recognizing that the graph is a vertical compression of the graph of the parent cubic function by a factor of 0.72. Then draw the horizontal line $m = 23$ and estimate the value of $\ell$ where the graphs intersect.

The graphs intersect where $\ell \approx 3.2$, so the edge length of the child’s block is about 3.2 cm.
Estimate the radius of a steel ball bearing with a mass of 75 grams and a density of 7.82 g/cm³.

Let \( r \) represent the radius (in centimeters) of the ball bearing. The volume \( V \) (in cubic centimeters) of the ball bearing is

\[
V(r) = \boxed{r^3}. \text{ The mass } m \text{ (in grams) of the ball bearing is } m(r) = \boxed{7.82 \cdot V(r) = \boxed{r^3}}.
\]

<table>
<thead>
<tr>
<th>Radius (cm)</th>
<th>Mass (g)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>1.5</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Draw the graph of the mass function, recognizing that the graph is a vertical ______ of the graph of the parent cubic function by a factor of ______. Then draw the horizontal line \( m = \boxed{\text{value}} \) and estimate the value of \( r \) where the graphs intersect.

The graphs intersect where \( r \approx \boxed{\text{value}} \), so the radius of the steel ball bearing is about \( \boxed{\text{value}} \) cm.

**Reflect**

7. **Discussion** Why is it important to plot multiple points on the graph of the volume function.
Your Turn

Use a cubic function to model the situation, and graph the function using calculated values of the function. Then use the graph to obtain the indicated estimate.

8. Polystyrene beads fill a cube-shaped box with an effective density of 0.00076 kg/cm³ (which accounts for the space between the beads). The filled box weighs 6 kilograms while the empty box had weighed 1.5 kilograms. Estimate the inner edge length of the box.

Elaborate

9. Identify which transformations (stretches or compressions, reflections, and translations) of \( f(x) = x^3 \) change the following attributes of the function.

   a. End behavior

   b. Location of the point of symmetry

   c. Symmetry about a point

10. Essential Question Check-In Describe the transformations you must perform on the graph of \( f(x) = x^3 \) to obtain the graph of \( g(x) = a(x - h)^3 + k \).
1. Graph the parent cubic function \( f(x) = x^3 \) and use the graph to answer each question.
   a. State the function's domain and range.
      The domain is \( \mathbb{R} \), and the range is \( \mathbb{R} \).
   b. Identify the function's end behavior.
      As \( x \to +\infty \), \( f(x) \to +\infty \). As \( x \to -\infty \), \( f(x) \to -\infty \).
   c. Identify the graph's x- and y-intercepts.
   The graph's only x-intercept is 0. The graph's only y-intercept is 0.
   d. Identify the intervals where the function has positive values and where it has negative values.
      The function has positive values on the interval \( (0, +\infty) \) and negative values on the interval \( (-\infty, 0) \).
   e. Identify the intervals where the function is increasing and where it is decreasing.
      The function is increasing throughout its domain.
      The function never decreases.
   f. Tell whether the function is even, odd, or neither. Explain.
      The function is odd because \( f(-x) = (-x)^3 = -x^3 = -f(x) \).
   g. Describe the graph's symmetry.
      The graph is symmetric about the origin.

Describe how the graph of \( g(x) \) is related to the graph of \( f(x) = x^3 \).

2. \( g(x) = (x - 4)^3 \)
   Translation of the graph of \( f(x) \) right 4 units.

3. \( g(x) = -5x^3 \)
   Vertical stretch of the graph of \( f(x) \) by a factor of 5 and a reflection across the x-axis.

4. \( g(x) = x^3 + 2 \)
   Translation of the graph of \( f(x) \) up 2 units.

5. \( g(x) = (3x)^3 \)
   Horizontal compression of the graph of \( f(x) \) by a factor of \( \frac{1}{3} \).
6. \( g(x) = (x + 1)^3 \)

7. \( g(x) = \frac{1}{4}x^3 \)

8. \( g(x) = x^3 - 3 \)

9. \( g(x) = \left( \frac{2}{3} \right)^x \)

Identify the transformations of the graph of \( f(x) = x^3 \) that produce the graph of the given function \( g(x) \). Then graph \( g(x) \) on the same coordinate plane as the graph of \( f(x) \) by applying the transformations to the reference points \((-1, -1), (0, 0), \text{ and } (1, 1)\).

10. \( g(x) = \frac{1}{2}x^3 \)

11. \( g(x) = \frac{1}{3}x^3 \)

12. \( g(x) = (x - 4)^3 - 3 \)

13. \( g(x) = (x + 1)^3 + 2 \)
A general equation for a cubic function $g(x)$ is given along with the function’s graph. Write a specific equation by identifying the values of the parameters from the reference points shown on the graph.

14. $g(x) = \left(\frac{1}{b}(x - h)^3 + k\right) - 3$

15. $g(x) = a(x - h)^3 + k$

16. $g(x) = \left(\frac{1}{b}(x - h)^3 + k\right) - 1$

17. $g(x) = a(x - h)^3 + k$
Use a cubic function to model the situation, and graph the function using calculated values of the function. Then use the graph to obtain the indicated estimate.

18. Estimate the edge length of a cube of gold with a mass of 1 kg. The density of gold is 0.019 kg/cm³.

19. A proposed design for a habitable Mars colony is a hemispherical biodome used to maintain a breathable atmosphere for the colonists. Estimate the radius of the biodome if it is required to contain 5.5 billion cubic feet of air.
20. **Multiple Response** Select the transformations of the graph of the parent cubic function that result in the graph of \( g(x) = (\frac{1}{3}(x - 3)^3 - 2)^3 + 1 \).

- A. Horizontal stretch by a factor of 3
- B. Horizontal compression by a factor of \( \frac{1}{3} \)
- C. Vertical stretch by a factor of 3
- D. Vertical compression by a factor of \( \frac{1}{3} \)
- E. Translation 1 unit up
- F. Translation 1 unit down
- G. Translation 2 units left
- H. Translation 2 units right

**H.O.T. Focus on Higher Order Thinking**

21. **Justify Reasoning** Explain how horizontally stretching (or compressing) the graph of \( f(x) = x^3 \) by a factor of \( b \) can be equivalent to vertically compressing (or stretching) the graph of \( f(x) = x^3 \) by a factor of \( a \).

22. **Critique Reasoning** A student reasoned that \( g(x) = (x - h)^3 \) can be rewritten as \( g(x) = x^3 - h^3 \), so a horizontal translation of \( h \) units is equivalent to a vertical translation of \( -h^3 \) units. Is the student correct? Explain.