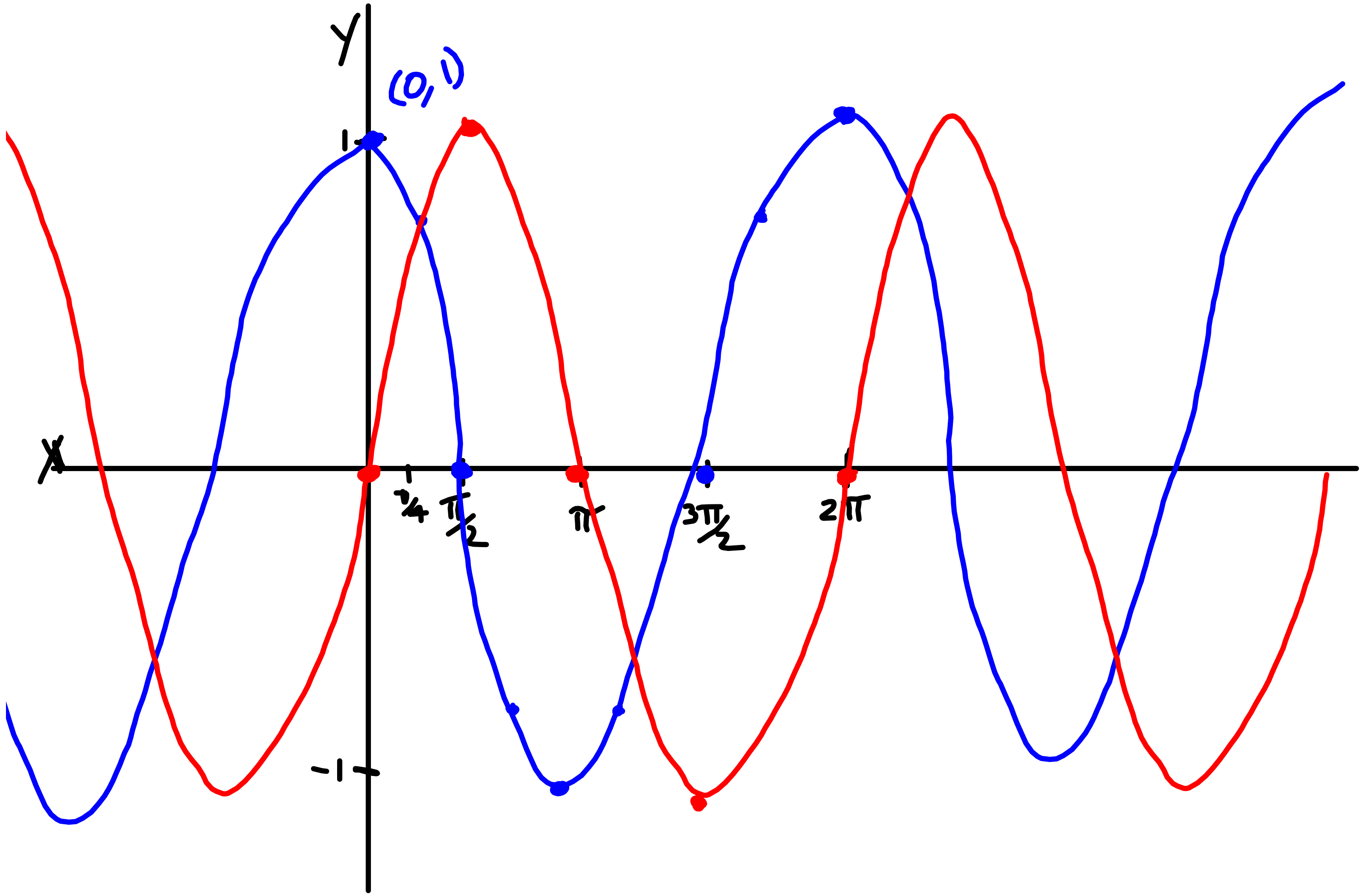
$y = \cos x$ $y = \sin x$

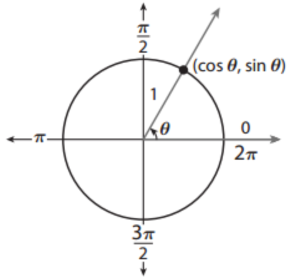


18.1 Stretching, Compressing, and Reflecting Sine and Cosine Graphs

Essential Question: What are the key features of the graphs of the sine and cosine functions?

Explore 1 Graphing the Basic Sine and Cosine Functions

Recall that the points around the unit circle have coordinates $(\cos\theta, \sin\theta)$ as shown.



(A) Identify the following points on the graph of the sine function on the interval $[0, 2\pi]$.

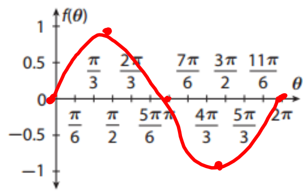
A. the three points where x-intercepts occur

B. the point of maximum value

C. the point of minimum value

Company

(B) Complete the table of values. Plot the points from the table, and draw a smooth curve through them.



θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$f(\theta) = \sin\theta$	0		1		0		-1		0

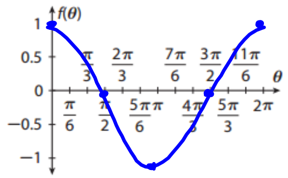
Ⓒ Identify the following points on the graph of the cosine function on the interval $[0, 2\pi]$.

A. the two points where x-intercepts occur

B. the two points of maximum value

C. the point of minimum value

Ⓓ Complete the table of values. Plot the points from the table, and draw a smooth curve through them.



θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
$f(\theta) = \cos\theta$	1		0		−1		0		1

Reflect

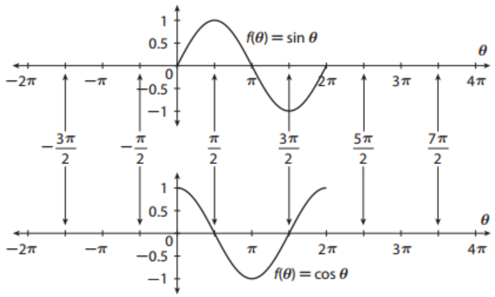
1. Give a decimal approximation of $\sin\frac{\pi}{3}$. Check to see whether the curve that you drew passes through the point $\left(\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$. What other points can you check based on the labeling of the θ–axis?

2. On the interval $0 \leq \theta \leq 2\pi$, where does the sine function have positive values? Where does it have negative values? Answer the same questions for cosine.

3. What are the minimum and maximum values of $f(\theta) = \sin\theta$ and $f(\theta) = \cos\theta$ on the interval $0 \leq \theta \leq 2\pi$? Where do the extreme values occur in relation to the θ–intercepts?

4. Describe a rotation that will map the graph of $f(\theta) = \sin\theta$ onto itself on the interval $0 \leq \theta \leq 2\pi$.

5. Recall that coterminal angles differ by a multiple of 2π and have the same sine value and the same cosine value. This means that the graphs of sine and cosine on the interval $0 \le \theta \le 2\pi$ represent one *cycle* of the complete graphs and that the cycles repeat every 2π radians. Use this fact to extend the graphs of $f(\theta) = \sin \theta$ and $f(\theta) = \cos \theta$ to the left and right by 1 cycle.



Explore 2 Graphing the Reciprocals of the Basic Sine and Cosine Functions

The **cosecant** and **secant** functions are the reciprocals of the sine and cosine functions, respectively.

$$\csc \theta = \frac{1}{\sin \theta} \qquad \sec \theta = \frac{1}{\cos \theta}$$

(A) Complete the table of values. Note that whenever $\sin \theta = 0$, $\csc \theta$ is undefined.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$f(\theta) = \sin \theta$	0	0.5	1	0.5	0	−0.5	−1	−0.5	0
$f(\theta) = \csc \theta$									

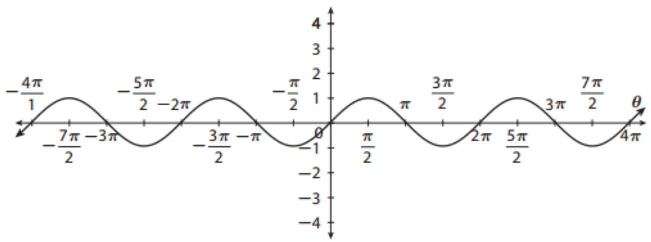
(B) Complete each of the following statements.

A. As $\theta \to 0^+$, $\sin \theta \to$ ☐ and $\csc \theta \to$ ☐.

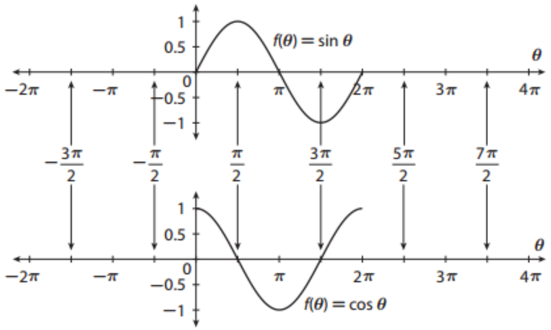
B. As $\theta \to 0^-$, $\sin \theta \to$ ☐ and $\csc \theta \to$ ☐.

What does this behavior tell you about the graph of the cosecant function?

(C) Sketch the graph of $f(\theta) = \csc \theta$ over the interval $[0, 2\pi]$. Then, extend the graph to the left and right until the entire coordinate plane is filled. Note that the sine function has been plotted for ease of graphing.



5. Recall that coterminal angles differ by a multiple of 2π and have the same sine value and the same cosine value. This means that the graphs of sine and cosine on the interval $0 \le \theta \le 2\pi$ represent one *cycle* of the complete graphs and that the cycles repeat every 2π radians. Use this fact to extend the graphs of $f(\theta) = \sin\theta$ and $f(\theta) = \cos\theta$ to the left and right by 1 cycle.



Explore 2 Graphing the Reciprocals of the Basic Sine and Cosine Functions

The **cosecant** and **secant** functions are the reciprocals of the sine and cosine functions, respectively.

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

(A) Complete the table of values. Note that whenever $\sin\theta = 0$, $\csc\theta$ is undefined.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{3\pi}{2}$	$\frac{11\pi}{6}$	2π
$f(\theta) = \sin\theta$	0	0.5	1	0.5	0	−0.5	−1	−0.5	0
$f(\theta) = \csc\theta$									

(B) Complete each of the following statements.

A. As $\theta \to 0^+$, $\sin\theta \to$ ☐ and $\csc\theta \to$ ☐.

B. As $\theta \to 0^-$, $\sin\theta \to$ ☐ and $\csc\theta \to$ ☐.

What does this behavior tell you about the graph of the cosecant function?

7. Describe the vertical asymptotes of the cosecant and secant functions over the interval $[0, 2\pi]$.

Vert stretch

Explain 1 Graphing $f(x) = a \sin \frac{1}{b} x$ or $f(x) = a \cos \frac{1}{b} x$

HORIZ STRETCH

In Explore 1, you graphed the sine and cosine functions on the interval $0 \le \theta \le 2\pi$, which represents all of the angles of rotation within the first counterclockwise revolution that starts at 0. Your drawings are not the complete graphs, however. They are simply one *cycle* of the graphs.

The graphs of sine and cosine consist of repeated cycles that form a wave-like shape. When a function repeats its values over regular intervals on the horizontal axis as the sine and cosine functions do, the function is called **periodic**, and the length of the interval is called the function's **period**. In Explore 1, you saw that the basic sine and cosine functions each have a period of 2π.

The wave-like shape of the sine and cosine functions has a "crest" (where the function's maximum value occurs) and a "trough" (where the function's minimum value occurs). Halfway between the "crest" and the "trough" is the graph's **midline**. The distance that the "crest" rises above the midline or the distance that the "trough" falls below the midline is called the graph's **amplitude**. In Explore 1, you saw that the basic sine and cosine functions each have an amplitude of 1.

Note that for trigonometric functions, the angle θ is the independent variable, and the output $f(\theta)$ is the dependent variable. You can graph these functions on the familiar xy-coordinate plane by letting x represent the angle and y represent the value of the function.

Example 1 **For each trigonometric function, identify the vertical stretch or compression and the horizontal stretch or compression. Then, graph the function and identify its period.**

(A) $y = 3 \sin 2x$

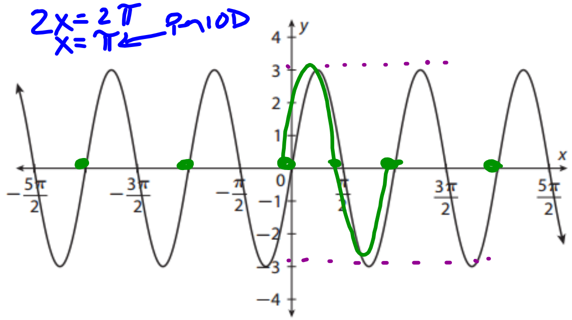
The equation has the general form $y = a \sin \frac{1}{b} x$. The value of a is 3. Since $\frac{1}{b} = 2$, the value of b is $\frac{1}{2}$. So, the graph of the parent sine function must be vertically stretched by a factor of 3 and horizontally compressed by a factor of $\frac{1}{2}$.

VERT STRETCH 3

HORIZ STRETCH

$2x = 2\pi$

$x = \pi \leftarrow$ PERIOD



Horizontally stretching or compressing the parent function's graph by a factor of $|b|$ changes the period of the function. Since the parent function has a period of 2π, multiply 2π by $|b|$ to obtain the period of the transformed function.

Period: $2\pi \cdot |b| = 2\pi \cdot \frac{1}{2} = \pi$

(B) $y = -3\cos\frac{x}{2}$ $\frac{x}{2} = 2\pi$ $x = 4\pi$

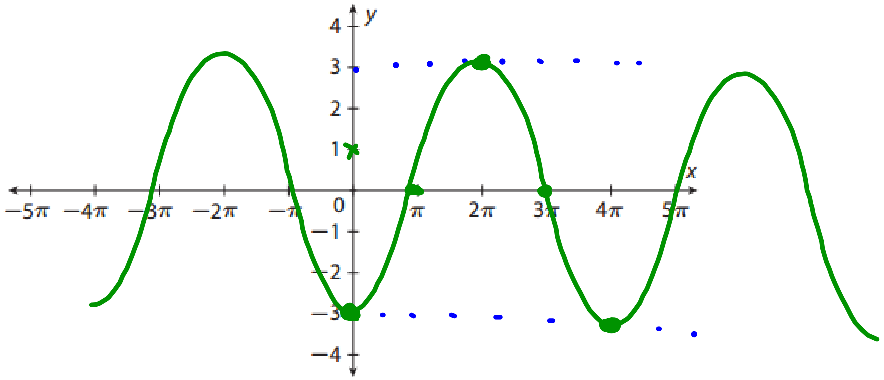
The equation has the general form $y = a\cos\frac{1}{b}x$. The value of a is -3. Since

$\frac{1}{b} =$ ☐, the value of b is ______.

So, the graph of the parent function must be vertically [stretched/compressed] by a factor of ______ and horizontally

[compressed/stretched] by a factor of ______.

Graph the function. Note that since a is negative, the graph will be reflected across the [x/y]-axis.



Find the function's period.

$2\pi \cdot |b| = 2\pi \cdot$ ☐ $=$ ☐

Reflect

8. For the function $y = a\sin\frac{1}{b}x$ or $y = a\cos\frac{1}{b}x$, what is the amplitude, under what circumstances is the graph of the function reflected about the x-axis, and how is the period determined?

____

____

____

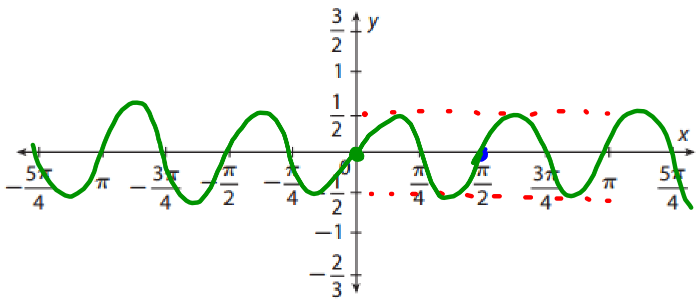
Your Turn

Identify the vertical stretch or compression and the horizontal stretch or compression. Then, graph the function and identify its period.

9. $y = \frac{1}{2}\sin 4x$

$4x = 2\pi$

$x = \frac{\pi}{2}$



Explain 2 Writing $f(x) = a \sin\frac{1}{b}x$ or $f(x) = a \cos\frac{1}{b}x$

You can write the equation of a trigonometric function if you are given its graph.

Example 2 **Write an equation for each graph.**

(A) Because the graph's y-intercept is 0, the graph is a sine function.

Since the maximum and minimum values are 2 and −2, respectively, the graph is a vertical stretch of the parent sine function by a factor of 2. So, $a = 2$.

The period of the function is 2.

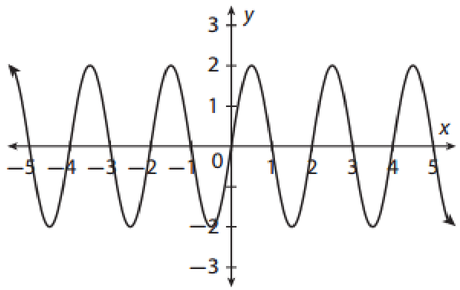
Use the equation $2\pi b = 2$ to find a positive value for $\frac{1}{b}$.

$$2\pi b = 2$$

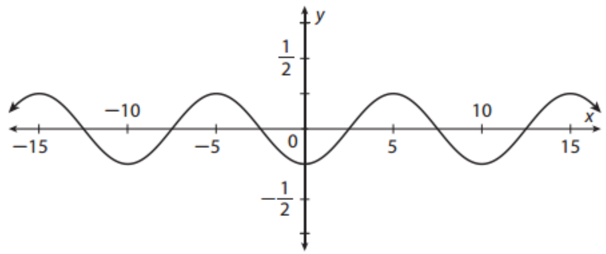
$$b = \frac{2}{2\pi} = \frac{1}{\pi}$$

$$\frac{1}{b} = \pi$$

An equation for the graph is $y = 2 \sin \pi x$.



B



Because the graph's y-intercept is negative, the graph is a[sine/cosine]function reflected across the x-axis.

Since the maximum and minimum in the graph are $\square$ and $\square$, respectively, the graph will be vertical [stretch/compression] of the graph of the parent cosine function by a factor of $\square$.

The period of the function is $\square$.

Use the equation $2\pi b = \square$ to find a positive value for $\frac{1}{b}$.

$2\pi b = \square$

$b = \frac{\square}{2\pi} = \frac{\square}{\pi}$

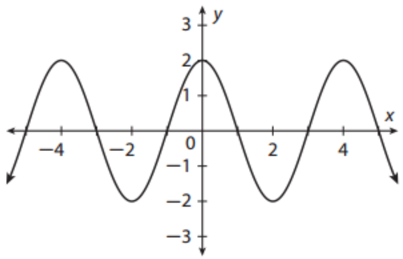
$\frac{1}{b} = \frac{\pi}{\square}$

An equation for the graph is $y = \square \cos \square x$.

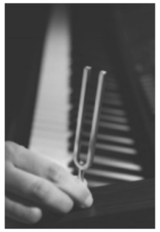
Your Turn

Write an equation for the graph.

10.



Explain 3 Modeling with Sine or Cosine Functions



Sine and cosine functions can be used to model real-world phenomena, such as sound waves. Different sounds create different waves. One way to distinguish sounds is to measure *frequency*. **Frequency** is the number of cycles in a given unit of time, so it is the reciprocal of the period of a function.

Hertz (Hz) is the standard measure of frequency and represents one cycle per second. For example, the sound wave made by a tuning fork for middle A has a frequency of 440 Hz. This means that the wave repeats 440 times in 1 second.

As a tuning fork vibrates, it creates fluctuations in air pressure. The maximum change in air pressure, typically measured in pascals, is the sound wave's amplitude.

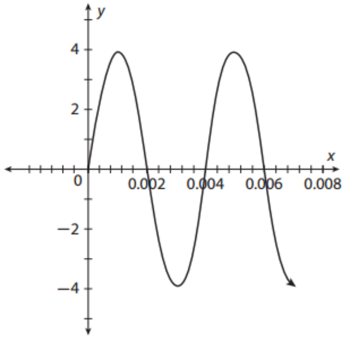
Example 3 **Graph each function, and then find its frequency. What do the frequency, amplitude, and period represent in the context of the problem?**

(A) **Physics** Use a sine function to graph a sound wave with a period of 0.004 second and an amplitude of 4 pascals.

Graph the function.

frequency $= \frac{1}{\text{period}} = \frac{1}{0.004} = 250$ Hz

The frequency represents the number of cycles of the sound wave every second. The amplitude represents the maximum change in air pressure. The period represents the amount of time it takes for the sound wave to repeat.

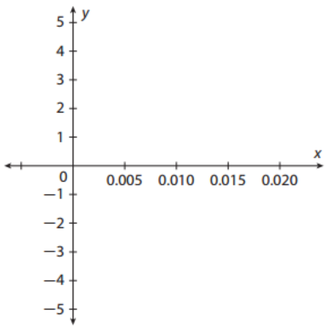


(B) **Physics** Use a cosine function to graph a sound wave with a period of 0.010 second and an amplitude of 3 pascals. Note that the recording of the sound wave started when the wave was at its maximum height.

Graph the function.

frequency $= \frac{1}{\text{period}} = \frac{1}{\boxed{}} = \boxed{}$ Hz

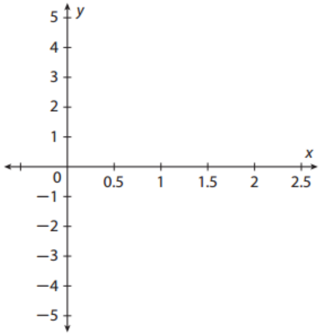
The frequency represents the number cycles of the sound wave every ________. The amplitude represents the maximum change in ____________. The period represents the amount of time it takes for the sound wave to [end/repeat].



Your Turn

Graph the function, and then find its frequency. What do the frequency, amplitude, and period represent in the context of the problem?

11. A pendulum makes one back-and-forth swing every 1.5 seconds. Its horizontal displacement relative to its position at rest is measured in inches. Starting when the pendulum is 5 inches (its maximum displacement) to the right of its position at rest, use a cosine function to graph the pendulum's horizontal displacement over time.



Elaborate

12. The graphs of sine and cosine are periodic functions. Refer to angles of rotation and the unit circle to explain why this is so.

13. Referring to angles of rotation and the unit circle, explain why $\sin(-\theta) = -\sin\theta$, and why $\cos(-\theta) = \cos\theta$. Use the graphs of $\sin\theta$ and $\cos\theta$ for reference.

14. How does the unit circle explain why the 2 in $y = \sin 2x$ results in a horizontal compression of the graph of $y = \sin x$?

15. **Essential Question Check-In** What is one key difference between the graphs of the sine and cosine functions?

Evaluate: Homework and Practice

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- Online Homework
- Hints and Help
- Extra Practice

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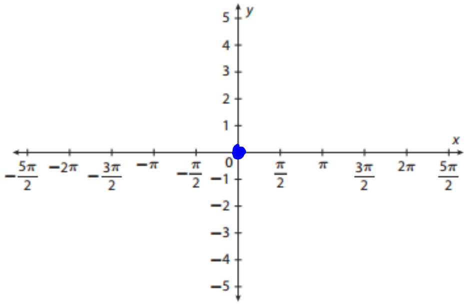
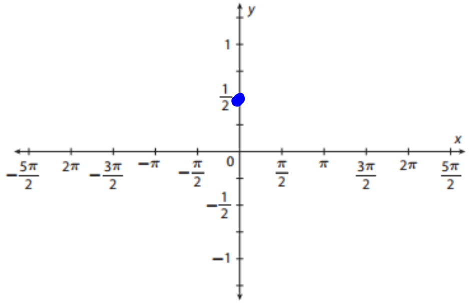
For each trigonometric function, identify the vertical stretch or compression and the horizontal stretch or compression. Then, graph the function and identify its period.

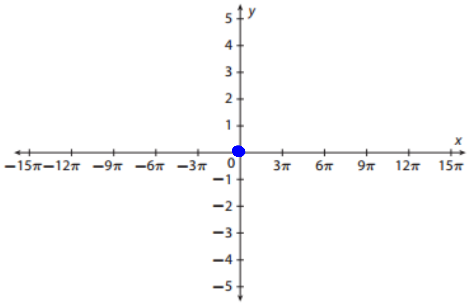
1. $y = 4 \sin x$



2. $y = \frac{1}{2} \cos 2x$



3. $y = -3 \sin \frac{1}{6}x$



4. $y = -2 \cos \frac{1}{3}x$